A characterization of semisimple local system by tame pure imaginary pluri-harmonic metric

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Abstract

Let L be a local system on a smooth quasi projective variety over $\boldsymbol{C}$. We see that L is semisimple if and only if there exists a tame pure imaginary pluri-harmonic metric on L. Although it is a rather minor refinement of a result of Jost and Zuo, it is significant for the study of harmonic bundles and pure twistor D-modules. As one of the application, we show that the semisimplicity of local systems are preserved by the pull back via a morphism of quasi projective varieties.

This paper will be included as an appendix to our previous paper, "*Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules*", math.DG/0312230.

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1 Introduction

1.1 Main result

1.1.1 Main theorem

Let X be a smooth projective variety over $\boldsymbol{C}$, and D be a normal crossing divisor of X. Let (E,∇) be a flat bundle on $X-D$. It is our main purpose to show the following theorem, which gives a characterization of semisimplicity of flat bundles by the existence of a pure imaginary pluri-harmonic metric.

Theorem 1.1

- *The flat bundle (E,∇) is semisimple if and only if there exists a tame pure imaginary pluri-harmonic metric h on (E,∇).*
- *If (E,∇) is simple, then the tame pure imaginary pluri-harmonic metric h is uniquely determined up to positive constant multiplication.*

Let us explain a tame pure imaginary pluri-harmonic metric. From a pluri-harmonic metric h on (E,∇), we obtain the harmonic bundle $(E,\overline{\partial}_E,\theta,h)$. Roughly speaking, $(E,\overline{\partial}_E,\theta,h)$ is tame if there exists a holomorphic bundle $\tilde{E}$ and a regular Higgs field $\tilde{\theta}\in End(\tilde{E})\otimes\Omega_X^{1,0}(\log D)$ such that $(\tilde{E},\tilde{\theta})_{|X-D}=(E,\theta)$. If any eigenvalues of the residues of $\tilde{\theta}$ are pure imaginary, then $(E,\overline{\partial}_E,\theta,h)$ is called pure imaginary. We remark that the eigenvalues of the residues are independent of a choice of a prolongment $(\tilde{E},\tilde{\theta})$.

1.1.2 Some remarks

It can be said that Theorem 1.1 is a partial refinement of the result of Jost-Zuo in [23], which is technically minor but significant for our application. Let us explain for more detail.

In [23], Jost and Zuo discussed the existence of tame pluri-harmonic (twisted) maps from a complement of a normal crossing divisor in a compact Kahler manifold to the symmetric space of non-compact types. On the other hand, we only consider $GL(r)/U(r)$ and a quasi projective variety as the target space and the domain respectively. However we can impose the pure imaginary condition to the behaviour of the tame pluri-harmonic twisted map at infinity. We can also derive the 'only if' part. It is the meaning of 'partial refinement'.

The refinement itself is rather technically minor in the following sense. We can observe 'the pure imaginary condition' easily from the argument in [23]. The 'only if' part is a rather easy consequence of the observation of Sabbah given in [35].

However, it seems significant for the theory of pure twistor D-modules. Briefly speaking, Theorem 1.1 gives a characterization of semisimple local system on a quasi projective variety. From Theorem 1.1, we can derive the correspondence of semisimple perverse sheaves and a 'pure imaginary' pure twistor D-modules, although the latter has not appeared in the literature. By using it together with Sabbah's theory [35] and our result in [33], it seems possible to show the regular holonomic version of Kashiwara's conjecture (see Introduction in [33]). We will discuss it elsewhere. In this paper, we give only the following theorem as an easy application, which is an affirmative answer to a question posed by Kashiwara.

Theorem 1.2 (Theorem 7.1) *Let X and Y be irreducible quasi projective varieties over $\boldsymbol{C}$. Let $F : X \longrightarrow Y$ be a morphism. Let L be a semisimple local system on Y. Then the pull back $F^{-1}(L)$ is also semisimple.* ∎

As is noted above, our main result (Theorem 1.1) can be regarded as a technically minor and partial refinement of the result of [23], once we completely understand the proof of the existence theorem of pluri-harmonic metric. Hence the author should explain why he writes this paper, which looks rather long. He feels that the argument of Jost-Zuo seems the argument for the experts of harmonic maps, and that it may not seem so easy to understand for non-specialists, at a sight. However, Theorem 1.1 is one of the most important key steps in our application of harmonic bundle to pure twistor D-modules. Hence the author thinks it appropriate to give a detailed proof of Theorem 1.1, which is available for a wide range of the readers. We will start from elementary facts, and we give the proofs of some rather well known results when the author does not know an appropriate reference. We give the details of the estimates, because it is rather delicate to deal with the infinite energy. We have to control the divergent term carefully. It is one of the reason why the paper is rather long.

As is explained, the most part of the paper is an effect of our effort to understand [23], and the most essential ideas for the existence part of Theorem 1.1 are due to Jost and Zuo, although we do not follow their arguments straightforwardly. Needless to say, the author is responsible for any mistakes contained in this paper.

1.2 The outline of the paper

1.2.1 Section 2

In the subsection 2.2, we recall some standard facts just for our reference of the later discussion. In the subsection 2.3, we recall the elementary geometry of the symmetric space $\mathcal{PH}(r)$ of the positive definite hermitian metrics. Lemma 2.15 is one of the key lemmas, although it is elementary. We also give the comparison of the distance of the hermitian metrics in $\mathcal{PH}(r)$ and the norm of the identity with respect to the two metrics in the subsubsection 2.3.5.

In the subsection 2.4, we discuss a twisted map associated with a commuting tuple of endomorphisms. The result will be used in the subsection 5.2 to construct the twisted map of $(\overline{\Delta}^*)^2$ whose energy is controlled.

In the subsection 2.5, we recall some standard facts on twisted harmonic maps. The Bochner type formula in the subsubsection 2.5.5 is due to Corlette. A variation of Bochner type formula is given in the subsubsection 2.5.6. They will be used in the proof of the pluri-harmonicity (see the subsections 6.2–6.3). It is important for our argument to consider two kinds of Bochner type formula.

1.2.2 Section 3

In the section 3, we give the definition of tame pure imaginary harmonic bundles, and some of useful properties. In the subsection 3.1, we give a definition of pure imaginary property of tame harmonic bundles. In the subsection 3.2, we see the estimate of the energy functions of a tame pure imaginary harmonic bundles on a punctured disc. We give a characterization of tame and pure imaginary properties by an increasing order of the energy in the subsubsection 3.2.3.

In the subsection 3.3, we show that the underlying flat bundle of a tame pure imaginary harmonic bundle is semisimple. It is a consequence of the observation, which is essentially due to Sabbah.

In the subsection 3.4, we see the maximum principle for the distance of two tame pure imaginary harmonic metrics on a punctured disc. The result will be useful to control the energy (the section 5). It will be also used to show the tameness (the subsection 6.4).

In the subsection 3.5, we show the uniqueness of tame pure imaginary harmonic bundle of a flat bundle on a quasi projective variety. Note that it essentially follows from the Kobayashi-Hitchin correspondence of Simpson and Biquard ([39] and [2]). However the detailed proof for uniqueness for in the case of parabolic flat bundle seems to be omitted there. Thus we give the detailed proof within our necessity. We essentially follow the argument of Corlette.

1.2.3 Section 4

In the subsection 4.1, we discuss the Dirichlet problem of a tame pure imaginary harmonic bundle on a punctured disc. The argument seems essentially due to Lohkamp [29] and Jost-Zuo [23].

In the subsection 4.2, we discuss the family version of the Dirichlet problem. We give the estimate of the differentials, which is given by the maximum principle (the subsection 3.4) and the estimate in the subsection 3.2. The idea is essentially due to Jost and Zuo ([22]). The result will be used for the construction of the twisted map whose energy is controlled (the section 5).

1.2.4 Section 5

We construct the twisted map on the complement of a normal crossing divisor in a compact Kahler manifold. We essentially follow the method of Jost-Zuo ([22] and [23]).

In the subsection 5.1, we construct the twisted map around smooth points of a divisor, by solving the family of the Dirichlet problem. Then the energy of the map is controlled by the result of the subsection 4.2. We also give the lower bound of the energy for arbitrary twisted map around smooth points of a divisor in the subsubsection 5.1.5. Essentially it is a consequence of Lemma 2.15. However we need some care to control the divergent term precisely.

In the subsection 5.2, we construct the twisted map around the intersection of the divisors. The results in the subsection 2.4 and the subsubsection 4.2.2 are used. We also give the lower bound of the energy of arbitrary twisted maps. Again, we have to be careful to control the divergent term.

In the subsection 5.3, we give the decomposition of the complement of a normal crossing divisor in a compact Kahler manifold, and we obtain the twisted map whose energy is controlled. We also give the lower bound of the energy of arbitrary twisted map. They are direct consequences of the results in the subsections 5.1–5.2.

1.2.5 Section 6

In the subsection 6.1, we obtain the harmonic metric of a semisimple flat bundle on a quasi projective variety. The argument is essentially same as that in the Dirichlet problem on a punctured disc (the subsection 4.1), except for the use of the argument in [21]. In the subsubsections 6.1.2–6.1.4, we give the detailed estimate of the energy of the resulted harmonic metric. They are necessary for the later discussion.

In the subsection 6.2, we show that $\overline{\partial}\theta$ and θ^2 are L^2, where θ is the associated $(1,0)$-form for the resulted harmonic metric. We use the Bochner type formula in the subsubsection 2.5.5. If the integral of the Bochner type formula would vanish, then we would obtain the pluri-harmonicity. However it does not seem easy to show such vanishing directly. (See the convergence (81), for example.)

Hence, in the subsection 6.3, we use another kind of Bochner type formula given in the subsubsection 2.5.6. It is rather easy to show the vanishing of the integral in this time, by using the L^2-property of $\overline{\partial}\theta$ obtained in the subsection 6.2. As a result, we obtain the pluri-harmonicity of the harmonic metric obtained in the subsection 6.1.

In the subsection 6.4, we show the tameness and the pure imaginary property for the resulted pluri-harmonic metric. It is rather easy consequence of the estimate of the energy given in the subsubsection 6.1.4, the characterization of tame pure imaginary harmonic bundle on a punctured disc (the subsubsection 3.2.3), the maximum principle (the subsection 3.5) and Hartogs type theorem.

Thus we obtain the existence theorem of a tame and pure imaginary pluri-harmonic harmonic metric for any semisimple flat bundle on a quasi projective surface, or more generally, the complement of a normal crossing divisor in a compact Kahler surface.

In the subsection 6.5, we show the existence of tame and pure imaginary pluri-harmonic metric in the higher dimensional case, by reducing the problem to the case of quasi projective surface. Here we need the quasi projectivity. To generalize the argument in the sections 5–6 in higher dimensional case, it seems that we need some additional arguments. For example, naively speaking, we need the family version of the argument in the subsection 5.2. However, perhaps, the author feels that it seems not so straightforward, for we cannot use the maximum principle for the family version of the map F on Y, for example.

1.2.6 Section 7

As a simple application, we show Theorem 7.1. We have only to show that the pull back of a tame pure imaginary harmonic bundle is also a tame pure imaginary harmonic bundle.

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2 Preliminary

2.1 Notation

2.1.1 Sets

We will use the following notation:

$\mathbb{Z}$:	the set of the integers,	$\mathbb{Z}_{>0}$:	the set of the positive integers,
$\boldsymbol{Q}$:	the set of the rational numbers,	$\boldsymbol{Q}_{>0}$:	the set of the positive rational numbers,
$\boldsymbol{R}$:	the set of the real numbers,	$\boldsymbol{R}_{>0}$:	the set of the positive real numbers,
$\boldsymbol{C}$:	the set of the complex numbers,	$\underline{n}$:	the set $\{1, 2, \ldots, n\}$,
$M(r)$:	the set of $r \times r$-matrices,	$\mathcal{H}_r$:	the set of $r \times r$-hermitian matrices,

We denote the set of positive hermitian metric of V by $\mathcal{PH}(V)$. We often identify it with the set of the positive hermitian matrices by taking an appropriate base of V.

We put $[a,b] := \{x \in \boldsymbol{R} \,|\, a \leq x \leq b\}$, $[a,b[:= \{x \in \boldsymbol{R} \,|\, a \leq x < b\}$, $]a,b] := \{x \in \boldsymbol{R} \,|\, a < x \leq b\}$ for any $a, b \in \boldsymbol{R}$.

2.1.2 A disc, a punctured disc and some products

For any positive number $C > 0$ and $z_0 \in \boldsymbol{C}$, the open disc $\big\{z \in \boldsymbol{C} \,\big|\, |z - z_0| < C\big\}$ is denoted by $\Delta(z_0, C)$, and the punctured disc $\Delta(z_0, C) - \{z_0\}$ is denoted by $\Delta^*(z_0, C)$. When $z_0 = 0$, $\Delta(0, C)$ and $\Delta^*(0, C)$ are often denoted by $\Delta(C)$ and $\Delta^*(C)$. Moreover, if $C = 1$, $\Delta(1)$ and $\Delta^*(1)$ are often denoted by Δ and Δ^*. If we emphasize the variable, we describe as Δ_z, Δ_i. For example, $\Delta_z \times \Delta_w = \{(z, w) \in \Delta \times \Delta\}$, and $\Delta_1 \times \Delta_2 = \{(z_1, z_2) \in \Delta \times \Delta\}$. We often use the notation $\boldsymbol{C}_\lambda$ and $\boldsymbol{C}_\mu$ to denote the complex planes $\big\{\lambda \in \boldsymbol{C}\big\}$ and $\big\{\mu \in \boldsymbol{C}\big\}$.

Unfortunately, the notation Δ is also used to denote the Laplacian. The author hopes that there will be no confusion.

2.2 Miscellaneous

2.2.1 Differentiability of Lipschitz continuous functions

We use the coordinate $(x, y_1, \ldots, y_l)$ for $\boldsymbol{R} \times \boldsymbol{R}^l$.

Lemma 2.1 *Let f be a Lipschitz continuous function on $\boldsymbol{R} \times \boldsymbol{R}^l$. Then $\frac{\partial f}{\partial x}$ is defined almost everywhere. Namely we have the measurable function F such that the following holds almost everywhere:*

$$F(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

Moreover $\frac{\partial f}{\partial x}$ is bounded.

Proof In the case $l = 0$, the differentiability of an absolute continuous function, and thus a Lipschitz continuous function, is well known. Let us consider the general case. Let h_i be any sequence of real numbers such that $h_i \to 0$. We put $F_i := h_i^{-1} \cdot \big(f(x+h_i,y) - f(x,y)\big)$, and then we obtain the sequence of the measurable functions $\{F_i\}$. It is well known that $\overline{\lim} F_i$ and $\underline{\lim} F_i$ are measurable. Thus the set $S := \big\{(x,y) \,\big|\, \overline{\lim} F_i(x,y) \neq \underline{\lim} F_i(x,y)\big\}$ is measurable. By using the result in the case $l = 0$, we can easily derive that the measure of S is 0. Hence we obtain the measurable function $F := \lim F_i$. It is easy to check that F has the desired properties. The boundedness follows from $|f(x+h,y) - f(x,y)| \leq C \cdot h$ for some constant C. See any appropriate text book of measure theory for the facts we used in the argument. ∎

Let f be a Lipschitz continuous function on $\boldsymbol{R} \times \boldsymbol{R}^l$. We obtain the measurable function $\partial f / \partial x$, which is bounded. It naturally gives the distribution.

On the other hand, f naturally gives the distribution. Hence we obtain the differential of f with respect to the variable x as the distribution, which we denote by $D_x f$.

Lemma 2.2 *We have $D_x f = \partial f / \partial x$ as the distribution.*

Proof Let ϕ be a test function. We have the following equality:

$$\int_{\boldsymbol{R}^{l+1}} \frac{f(x+h,y) - f(x,y)}{h} \cdot \phi(x,y) = \int_{\boldsymbol{R}^{l+1}} f(x,y) \cdot \frac{\phi(x-h,y) - \phi(x,y)}{h}. \tag{1}$$

Since f is bounded, the right hand side of (1) converges to $-\int_{\boldsymbol{R}^{l+1}} f \cdot (\partial \phi / \partial x)$, due to the dominated convergence theorem. Since f is Lipschitz, there exists a positive constant C such that $\big|h^{-1} \cdot \big(f(x+h,y) - f(x,y)\big)\big| \leq C$ holds for any h. Hence the left hand side of (1) converges to $\int_{\boldsymbol{R}^{l+1}} (\partial f / \partial x) \cdot \phi$. Thus we are done. ∎

Corollary 2.1 *Let f be a Lipschitz function on $\boldsymbol{R}^l$. Then f is locally L_1^p for any real number $p \geq 1$.* ∎

2.2.2 Elementary linear algebra

Let V be an r-dimensional vector space, and h be a hermitian metric of V. Let $\boldsymbol{v}$ be a base of V. We put $H := H(h, \boldsymbol{v})$. Let f be an endomorphism of V, and then we have the matrix A such that $f \cdot \boldsymbol{v} = \boldsymbol{v} \cdot A$. Let $f^\dagger$ denote the adjoint of f with respect to the metric h. Then we have $f^\dagger \boldsymbol{v} = \boldsymbol{v} \cdot \overline{H}^{-1} \cdot {}^t\overline{A} \cdot \overline{H}$.

We have the norm $|f|_h$ of f with respect to the metric h.

Lemma 2.3 *The following holds:*

$$|f|_h^2 = \mathrm{tr}\big(A \cdot \overline{H}^{-1} \cdot {}^t\overline{A} \cdot \overline{H}\big).$$

Proof We have $|f|_h^2 = \mathrm{tr}(f \cdot f^\dagger)$. Then the claim immediately follows. ∎

2.2.3 Hartogs type theorem

Lemma 2.4 *Let F be a holomorphic function on $\overline{\Delta}^2 - \{z_2 = 0\}$. Assume that $F_{|\pi_2^{-1}(Q)}$ is holomorphic for almost every $Q \in \{z_2 = 0\}$. Then F is holomorphic on Δ.*

Proof We put as follows:

$$G(z_1, z_2) := \int_{|\zeta|=1} \frac{G(z_1, \zeta)}{(\zeta - z_2)} \cdot \frac{d\zeta}{2\pi\sqrt{-1}}.$$

Then $G(z_1, z_2)$ gives a holomorphic function on $\overline{\Delta}^2$. Due to the assumption, there exists a dense subset $Y \subset \overline{\Delta}^2 - \{z_2 = 0\}$ such that $G_{|Y} = F_{|Y}$. Hence we obtain $G = F$ on $\overline{\Delta}^2 - \{z_2 = 0\}$. ∎

The following lemma will be used later, which can be shown similarly.

Lemma 2.5 *Let F be a holomorphic function on $\overline{\Delta}^2 - \{z_2 = 0\}$. Assume that there exists an open subset $U \subset \{z_2 = 0\}$ such that $F_{|\pi_2^{-1}(Q)}$ is holomorphic on $Q \times \overline{\Delta}$. Then F is holomorphic on $\overline{\Delta}^2$.* ∎

2.2.4 Lefschetz hyperplane theorem for the fundamental group

Lemma 2.6 *Let X be a smooth projective surface, and D be a normal crossing divisor of X. Let H_1 be a sufficiently ample smooth divisor such that $H_1 \cup D$ is normal crossing. Then $\pi_1(H_1 \setminus D) \longrightarrow \pi_1(X - D)$ is surjective.*

Proof We give only a sketch of a proof. Let D_1 be an ample smooth divisor of X such that $D_0 = D_1 \cup D$ is very ample and normal crossing. Let $D = D_2 \cup \cdots \cup D_l$ be the irreducible decomposition. Let N_i denote the tubular neighbourhood of D_i $(i = 1, \ldots, l)$. We put $N_0 := \bigcup_{i=1}^l N_i$. Recall that we can decompose X into N_0 and i-handles $(i = 2, 3, 4)$ (See [31]). Hence the inclusion $N_0 - D \longrightarrow X - D$ induces the surjection $\pi_1(N_0 - D) \longrightarrow \pi_1(X - D)$.

Let s be a section of $\mathcal{O}(D_0)$ satisfying that the zero set $H_1 := s^{-1}(0)$ is smooth, the $H_1 \cap D_0$ is contained in the smooth part of D_0, and $H_1 \cup D$ is normal crossing.

We take the Lefschetz pencil X' of H_1 and D_0. We take the desingularization $\tilde{X}$ of X'. We have the birational morphism $p : \tilde{X} \longrightarrow X$ and the morphism $\pi : \tilde{X} \longrightarrow \mathbb{P}^1$. We may assume that $\pi^{-1}(0) = p^{-1}(D_0) = p^{-1}(D) \cup p^{-1}(D_1)$. For any sufficiently small $\epsilon > 0$ and for any point $t \in \Delta^*(\epsilon)$, $\pi^{-1}(t)$ is smooth.

Let us consider the inclusion $\iota_1 : \pi^{-1}(\Delta(\epsilon)) \setminus p^{-1}(D) \longrightarrow p^{-1}(N_0) \setminus p^{-1}(D)$. Let $P_1, \ldots, P_m$ denote the points of $H_1 \cap D$. We can take continuous maps $\varphi_i : S^1 \times S^1 \longrightarrow \partial\pi^{-1}(\Delta(\epsilon))$ such that $p^{-1}(N_0) \setminus p^{-1}(D)$ is homotopy equivalent to the topological space obtained from $\pi^{-1}(\Delta(\epsilon) - p^{-1}(D))$ and $\coprod_{i=1}^m S^1 \times D^2$ via the attaching maps φ_i $(i = 1, \ldots, m)$. Hence ι_1 induce the surjection $\pi_1\big(\pi^{-1}(\Delta(\epsilon)) \setminus p^{-1}(D)\big) \longrightarrow \pi_1\big(p^{-1}(N_0) \setminus p^{-1}(D)\big)$.

The fiber bundle $\pi^{-1}\big(\Delta^*(\epsilon)\big) \setminus p^{-1}(D_0) \longrightarrow \Delta^*(\epsilon)$. induces the exact sequence:

$$\pi_1\big(\pi^{-1}(t) \setminus p^{-1}(D_0)\big) \longrightarrow \pi_1\big(\pi^{-1}\big(\Delta^*(\epsilon)\big) \setminus p^{-1}(D_0)\big) \xrightarrow{\pi_*} \pi_1\big(\Delta^*(\epsilon)\big) \longrightarrow 1. \tag{2}$$

The inclusion $\iota_2 : \pi^{-1}\big(\Delta^*(\epsilon)\big) \setminus p^{-1}(D_0) \longrightarrow \pi^{-1}\big(\Delta(\epsilon)\big) \setminus p^{-1}(D)$ induces the surjection:

$$\iota_{2\,*} : \pi_1\big(\pi^{-1}(\Delta^*(\epsilon)) \setminus p^{-1}(D_0)\big) \longrightarrow \pi_1\big(\pi^{-1}(\Delta(\epsilon)) \setminus p^{-1}(D)\big). \tag{3}$$

We can take a loop γ around D_1 which is mapped to the generator of $\pi_1\big(\Delta^*(\epsilon)\big)$ via π_* in (2). On the other hand, γ is mapped to 0 via the map $\iota_{2\,*}$. Hence we can conclude that the inclusion $\pi^{-1}(t) \setminus p^{-1}(D_0) \longrightarrow \pi^{-1}(\Delta(\epsilon)) \setminus p^{-1}(D)$ induces the surjection of the fundamental groups.

In all, the natural morphism $\pi^{-1}(t) \setminus p^{-1}(D_0) \longrightarrow X - D$ induces the surjection of the fundamental groups. Since the morphism factors through $H_t \setminus D$, we obtain the surjectivity of $\pi_1\big(H_t \setminus D\big) \longrightarrow \pi_1(X - D)$ for any sufficiently small $t \neq 0$. Since we can isotopically deform $H_t \setminus D$ to $H_1 \setminus D$ in $X - D$, we can conclude that $\pi_1(H_1 \setminus D) \longrightarrow \pi_1(X - D)$ is surjective. ∎

Lemma 2.7 *Let X be a smooth projective variety, and D be a normal crossing divisor of X. If H is sufficiently ample smooth divisor such that $H \cup D$ is normal crossing, then the map $\pi_1(H \setminus D) \longrightarrow \pi_1(X \setminus D)$ is onto.*

Proof We give only a sketch of a proof. We use an induction on a dimension of X.

Let $D = \bigcup_{i=1}^l D_i$ be the irreducible decomposition. Let N_i be a tubular neighbourhood of D_i. Let N_H denote the tubular neighbourhood of H. We put $N = N_H \cup \big(\bigcup_{i=1}^l N_i\big)$. The morphism $\pi_1(N \setminus D) \longrightarrow \pi_1(X - D)$ is surjective [31].

Let γ be an element of $\pi_1(N \setminus D)$. Since each connected component $D_i \cap D_j$ intersects with H, we may decompose γ into the product of the paths γ_α such that they are represented by closed paths $\tilde{\gamma}_\alpha$ satisfying $\tilde{\gamma}_\alpha\big([0, a]\big) \subset N_H \setminus D$, $\tilde{\gamma}_\alpha\big([a, b]\big) \subset N_{i_\alpha} \setminus D$ and $\tilde{\gamma}_\alpha\big([b, 1]\big) \subset N_H \setminus D$.

Due to the hypothesis of our induction, the inclusion $D_{i_\alpha} \cap N_H \setminus \left(\bigcup_{j \neq i_\alpha} N_j\right) \subset D_{i_\alpha} \setminus \left(\bigcup_{j\neq i_\alpha} N_j\right)$ induces the surjection of the fundamental group. Since $N_{i_\alpha} \cap N_H \setminus \left(\bigcup_{j\neq i_\alpha} N_j\right)$ and $N_{i_\alpha} \cap \left(\bigcup_{j \neq i_\alpha} N_j\right)$ are disc bundles over $D_{i_\alpha} \cap N_H \setminus \left(\bigcup_{j\neq i_\alpha} N_j\right)$ and $D_{i_\alpha} \setminus \left(\bigcup_{j\neq i_\alpha} N_j\right)$ respectively. Thus it is easy to see that $\tilde{\gamma}_{i_\alpha}$ is homotopic to a closed path in $N_H \setminus D$. Hence we obtain the surjectivity desired. ∎

Remark 2.1 *In the proof, we have to take the tubular neighbourhoods N_i and N_H cleanly. We omit to give the detail.* ∎

Corollary 2.2 *Let X be a quasi projective variety. Let L be a local system on X. Let H be a sufficiently ample hypersurface of X. Then L is semisimple if and only if the restriction $L_{|H}$ is semisimple.* ∎

2.3 Elementary geometry of $GL(r)/U(r)$

2.3.1 The $GL(r)$-invariant metric

We have the standard left action κ of $GL(r)$ on $\mathcal{PH}(r)$:

$$GL(r) \times \mathcal{PH}(r) \longrightarrow \mathcal{PH}(r), \quad (g, H) \longmapsto \kappa(g, H) = g \cdot H \cdot {}^t\bar{g}.$$

For any point $H \in \mathcal{PH}(r)$, the tangent space $T_H\mathcal{PH}(r)$ is naturally identified with the vector space $\mathcal{H}(r)$. Let I_r denote the identity matrix. We have the positive definite metric of $T_{I_r}\mathcal{PH}(r)$ given by $(A, B)_{I_r} = \mathrm{tr}(A \cdot B) = \mathrm{tr}(A \cdot {}^t\bar{B})$. It is easy to see the metric is invariant with respect to the $U(r)$-action on $T_{I_r}\mathcal{PH}(r)$.

Let H be any point of $\mathcal{PH}(r)$ and let g be an element of $GL(r)$ such that $H = g \cdot {}^t\bar{g}$. Then the metric of $T_H\mathcal{PH}(r)$ is given as follows:

$$(A, B)_H = \big(\kappa(g^{-1})_* A, \kappa(g^{-1})_* B\big)_{I_r} = \mathrm{tr}\Big(g^{-1} A {}^t\bar{g}^{-1} \cdot g^{-1} B {}^t\bar{g}^{-1}\Big) = \mathrm{tr}\big(H^{-1} A H^{-1} B\big). \tag{4}$$

Since $(\cdot, \cdot)_{I_r}$ is $U(r)$-invariant, the metric $(\cdot, \cdot)_H$ on $T_H\mathcal{PH}(r)$ is well defined. Thus we have the $GL(r)$-invariant Riemannian metric of $\mathcal{PH}(r)$.

It is well known that $\mathcal{PH}(r)$ with the metric above is a symmetric space with non-positive curvature. We denote the induced distance by $d_{\mathcal{PH}(r)}$. We often use the simple notation d to denote $d_{\mathcal{PH}(r)}$, if there are no confusion.

Let X be a manifold, and $\Psi : X \longrightarrow \mathcal{PH}(r)$ be a differentiable map. Let P be a point of X, and v be an element of the tangent space $T_P X$.

Lemma 2.8 *We have the following formula:*

$$\big|d\Psi(v)\big|^2_{T_{\Psi(P)}\mathcal{PH}(r)} = \mathrm{tr}\Big(\Psi(P)^{-1} \cdot d\Psi(v) \cdot \Psi(P)^{-1} \cdot d\Psi(v)\Big) = \mathrm{tr}\Big(\overline{\Psi(P)}^{-1} \cdot \overline{d\Psi(v)} \cdot \overline{\Psi(P)}^{-1} \cdot \overline{d\Psi(v)}\Big).$$

Proof It follows from (4). ∎

2.3.2 The geodesics and some elementary estimates of the distances

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ be a tuple of real numbers. Let $\gamma_{\boldsymbol{\alpha}}(t)$ denotes the diagonal matrices whose (i, i)-th component is $e^{\alpha_i \cdot t}$. When we regard $\gamma_{\boldsymbol{\alpha}}$ as a path in $\mathcal{PH}(r)$, it is well known that $\gamma_{\boldsymbol{\alpha}}$ is a geodesic. We put $s := \sqrt{\sum \alpha_i^2} \cdot t$, and then $|s|$ gives the arc length from I_r. We put $\widetilde{\gamma}_{\boldsymbol{\alpha}}(s) := \gamma_{\boldsymbol{\alpha}}(t)$.

Let us consider the case $\alpha_1 > \alpha_2 > \cdots > \alpha_r$. Let k be an upper triangular matrices whose diagonal entries are 1. i.e., $k_{i\,j} = 0$ unless $i \leq j$ and $k_{i\,i} = 1$. The following lemma can be checked directly.

Lemma 2.9 *We put $C_2 := \min\big\{\alpha_i - \alpha_{i+1}\big\} > 0$. Then the following holds:*

$$\big|\big|I - \gamma_{\boldsymbol{\alpha}}(t)^{-1/2} \cdot k \cdot \gamma_{\boldsymbol{\alpha}}(t)^{1/2}\big|\big| \leq C_1 \cdot e^{-C_2 \cdot t}.$$

Here $||\cdot||$ denote the norm of $M(r)$. ∎

Lemma 2.10 *There exist positive numbers C_1 and C_2, independent of k, such that the following holds:*

$$d_{\mathcal{PH}(r)}\Big(\gamma_{\boldsymbol{\alpha}}(t),\,\kappa\big(k,\gamma_{\boldsymbol{\alpha}}(t)\big)\Big)\leq C_1\cdot e^{-C_2\cdot t}.$$

Proof We put $k(t):=\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot k\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2}$. We have the following:

$$d_{\mathcal{PH}(r)}\Big(\gamma_{\boldsymbol{\alpha}}(t),\,\kappa\big(k,\gamma_{\boldsymbol{\alpha}}(t)\big)\Big)=d_{\mathcal{PH}(r)}\Big(I_r,\,k(t)\cdot{}^t\overline{k(t)}\Big).$$

Then the claim immediately follows from Lemma 2.9. ∎

Lemma 2.11 *Let s, s_0 be non-negative numbers such that $s\geq s_0$. Let k be as above. Then we have the following inequality for some positive numbers C_1 and C_2 which are independent of k and s_0:*

$$\Big|d\Big(\widetilde{\gamma}_{\boldsymbol{\alpha}}(s),\,\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0)\big)\Big)-s+s_0\Big|\leq C_1\cdot e^{-C_2\cdot s}.$$

Proof We have the following triangle inequality:

$$\Big|d\Big(\widetilde{\gamma}_{\boldsymbol{\alpha}}(s),\,\kappa(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0))\Big)-d\Big(\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s)\big),\,\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0)\big)\Big)\Big|\leq d\Big(\widetilde{\gamma}_{\boldsymbol{\alpha}}(s),\,\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s)\big)\Big).$$

Due to Lemma 2.10, the right hand side is dominated by $C_1\cdot e^{-C_2 s}$ for some positive constants C_1 and C_2. On the other hand, we have $d\Big(\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s)\big),\,\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0)\big)\Big)=s-s_0$. Thus we are done. ∎

Let $B_{\widetilde{\gamma}_{\boldsymbol{\alpha}}}$ be the Busemann function given as follows (see [11] or [36], for example):

$$B_{\widetilde{\gamma}_{\boldsymbol{\alpha}}}(x):=\lim_{s\to\infty}\Big(d\big(\widetilde{\gamma}_{\boldsymbol{\alpha}}(s),x\big)-s\Big).$$

Then Lemma 2.11 is reformulated as follows:

$$B_{\widetilde{\gamma}_{\boldsymbol{\alpha}}}\big(\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0)\big)\big)=-s_0=B_{\widetilde{\gamma}_{\boldsymbol{\alpha}}}\big(\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0)\big). \tag{5}$$

Lemma 2.12 *We have the following inequality:*

$$d\big(I_r,\kappa\big(k,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0)\big)\big)\geq d\big(I_r,\widetilde{\gamma}_{\boldsymbol{\alpha}}(s_0)\big)=s_0.$$

Proof It follows from (5). Note that the horospheres and $\widetilde{\gamma}$ are orthogonal. See [36] for more detail. ∎

2.3.3 An estimate of infimum

Let A be an element of $GL(r)$. Let $a_1,\ldots,a_r$ be eigenvalues of A. We put as follows:

$$\rho(A):=\Big(\sum\big(\log|a_i|^2\big)^2\Big)^{1/2}.$$

Lemma 2.13 *Assume $|a_i|>|a_{i+1}|$ for any i. Then we have the following inequality, for any $H\in\mathcal{PH}(r)$:*

$$\rho(A)\leq d_{\mathcal{PH}(r)}\Big(\kappa(A,H),\,H\Big). \tag{6}$$

Proof We have only to show the inequality (6) in the case $H=I_r$ for any A. For any element $U\in U(r)$, we have the following:

$$d\Big(\kappa\big(UAU^{-1},I_r\big),\,I_r\Big)=d\Big(\kappa\big(A,I_r\big),\,I_r\Big),\quad \rho\big(UAU^{-1}\big)=\rho(A).$$

Hence we may assume that A is an upper triangular matrices such that whose (i,i)-th entries are a_i. Then we can decompose A into the product $A_1\cdot A_2$ such that the following holds:

- A_1 is the upper triangular matrix whose (i,i)-th entries are $|a_i|$.

- A_2 is the diagonal matrix such that the absolute values of the (i,i)-th entries are 1. In particular, A_2 is unitary.

Then it is easy to see that $\rho(A) = \rho(A_1)$ and $d\big(\kappa\big(A, I_r\big), I_r\big) = d\big(\kappa\big(A_1, I_r\big), I_r\big)$. Hence we may assume $a_i = e^{\alpha_i}$ for some real numbers α_i from the beginning. Note we have $\alpha_1 > \cdots > \alpha_r$, due to our assumption $|a_i| > |a_{i+1}|$ for any i.

We decompose A into the product $K \cdot A_0$ such that the following holds:

- K is the upper triangular matrix whose diagonal entries are 1.
- A_0 is the diagonal matrix whose (i,i)-th component is a_i.

Due to Lemma 2.12, we have the following inequality:

$$d\big(I,\, A\cdot{}^t\bar{A}\big) \geq d\big(I,\, A_0\cdot{}^t\bar{A}_0\big) = \Big(\sum \big(\log |a_i|^2\big)^2\Big)^{1/2} = \rho(A).$$

Thus we are done. ∎

Corollary 2.3 *Let A be any element of $GL(r)$. Then we have the inequality $\rho(A) \leq d\big(\kappa(A,H), H\big)$ for any $H \in \mathcal{PH}(r)$.*

Proof Let $a_1, \ldots, a_r$ be eigenvalues of A. In the case $|a_i| > |a_{i+1}|$, the claim is already shown in Lemma 2.13. Let us take a sequence $\{A^{(n)}\}$ in $GL(r)$ such that the following holds:

- The sequence $\{A^{(n)}\}$ converges to A.
- Let $a_1^{(n)}, \ldots, a_r^{(n)}$ be eigenvalues of $A^{(n)}$. Then the inequality $|a_i^{(n)}| > |a_{i+1}^{(n)}|$ hold for any i.

Then we have the inequalities $\rho(A^{(i)}) \leq d\big(\kappa(A^{(i)}, H), H\big)$. We also have the convergences $\rho(A^{(i)}) \longrightarrow \rho(A)$ and $d\big(\kappa(A^{(i)}, H), H\big) \longrightarrow d\big(\kappa(A, H), H\big)$. Thus we are done. ∎

Lemma 2.14 *Let A be any element of $GL(r)$. Then we have the following equality:*

$$\rho(A) = \inf\big\{d(\kappa(A,H), H) \,\big|\, H \in \mathcal{PH}(r)\big\}. \tag{7}$$

Proof We have already shown the inequality $\leq$ in (7) (Corollary 2.3). Let us show the inequality $\geq$. We may assume that A is an upper triangular matrix whose (i,i)-th entries are a_i. Let $\boldsymbol{\alpha}$ be any element of $\boldsymbol{R}^r$ such that $\alpha_i > \alpha_{i+1}$ for any i, and let us consider the geodesic $\gamma_{\boldsymbol{\alpha}}(t)$ (the subsubsection 2.3.2). We put $A(t) := \gamma_{\boldsymbol{\alpha}}(t)^{-1/2} \cdot A \cdot \gamma_{\boldsymbol{\alpha}}(t)^{1/2}$. Then $A(t)$ converges to the diagonal matrix C whose (i,i)-th entries are a_i. Since $A(t) \cdot {}^t\overline{A(t)}$ converges to $C \cdot \overline{C}$, we have the following convergence:

$$d\Big(\kappa\big(A, \gamma_{\boldsymbol{\alpha}}(t)\big),\, \gamma_{\boldsymbol{\alpha}}(t)\Big) = d\big(A(t)\cdot{}^t\overline{A(t)},\, I_r\big) \longrightarrow d\big(C \cdot \overline{C},\, I_r\big) = \rho(A).$$

Thus we are done. ∎

2.3.4 A lower bound of the energy

Let A be an element of $GL(r)$, and H be an element of $\mathcal{PH}(r)$. Let $\varphi : [0, 2\pi] \longrightarrow \mathcal{PH}(r)$ be an L_1^2-map such that $\varphi(0) = H$ and $\varphi(2\pi) = \kappa(A, H)$. Let g be a positive continuous function on $[0, 2\pi]$. The following inequality will often be used.

Lemma 2.15

$$\int_0^{2\pi} \left|\frac{\partial \varphi}{\partial t}\right|^2 \cdot g \cdot dt \geq \rho(A)^2 \cdot \left(\int g^{-1} dt\right)^{-1}.$$

Proof Due to the Schwarz's inequality, we have the following:

$$\left(\int_0^{2\pi} \left|\frac{\partial \varphi}{\partial t}\right|\right)^2 \leq \int_0^{2\pi} \left|\frac{\partial \varphi}{\partial t}\right|^2 \cdot g \cdot dt \times \int_0^{2\pi} g^{-1} \cdot dt.$$

Then the claim follows from Lemma 2.14. ∎

2.3.5 Comparison of the norm and the distance

For any elements H_1 and H_2 of $\mathcal{PH}(r)$, we have an element $g \in GL(r)$ such that $\kappa(g, H_1) = I_r$ and that $\kappa(g, H_2)$ is the diagonal matrix. The set of the eigenvalues of $\kappa(g, H_2)$ is independent of a choice of g. Let $e^{\alpha_1}, \ldots, e^{\alpha_r}$ be the eigenvalues of $\kappa(g, H_2)$. We put as follows:

$$\delta(H_1, H_2) := \left(\sum_{i=1}^{r} \Bigl(\frac{e^{\alpha_i} - e^{-\alpha_i}}{2} \Bigr)^2 \right)^{1/2}.$$

On the other hand, we have the distance $d_{\mathcal{PH}(r)}(H_1, H_2) = \bigl(\sum \alpha_i^2\bigr)^{1/2}$.

For any real number R, we put as follows:

$$C(R) := \frac{e^R - e^{-R}}{2R}.$$

If $0 \le x \le R$, we have $x \le C(R) \cdot x$. We also note that $C(R) \to 1$ when $R \to 0$.

Lemma 2.16

- *We have the inequality:*
$$d_{\mathcal{PH}(r)}(H_1, H_2) \le \delta(H_1, H_2).$$
- *If $d(H_1, H_2) \le R$, we have the inequality:*
$$\delta(H_1, H_2) \le C(R) \cdot d_{\mathcal{PH}(r)}(H_1, H_2).$$

Proof It can be checked elementarily. ∎

We reformulate Lemma 2.16 as follows: Let V be an r-dimensional vector space, and let h_1 and h_2 be hermitian metrics of V. The identity map induces the map $\Phi : (V, h_1) \longrightarrow (V, h_2)$. We have the norms $|\Phi|$ and $\bigl|\Phi^{-1}\bigr|$.

Lemma 2.17

- *The following inequality holds:*
$$d_{\mathcal{PH}(r)}(h_1, h_2)^2 \le \frac{\bigl|\Phi\bigr|^2 + \bigl|\Phi^{-1}\bigr|^2 - 2r}{4}.$$
- *If $d_{\mathcal{PH}(r)}(h_1, h_2) \le R$, the following inequality holds:*
$$\frac{\bigl|\Phi\bigr|^2 + \bigl|\Phi^{-1}\bigr|^2 - 2r}{4} \le C(R)^2 \cdot d_{\mathcal{PH}(r)}(h_1, h_2)^2.$$

Proof If we take an appropriate base of V, h_1 and h_2 are represented by the identity matrix I_r and the diagonal matrices whose diagonal entries are $e^{\alpha_1}, \ldots, e^{\alpha_r}$. It is easy to check that $|\Phi|^2 = \sum_{i=1}^{r} e^{2\alpha_i}$ and $|\Phi^{-1}|^2 = \sum_{i=1}^{r} e^{-2\alpha_i}$. Then it immediately follows $4^{-1}\bigl(|\Phi|^2 + |\Phi^{-1}|^2 - 2r\bigr) = \delta(H_1, H_2)^2$. Thus the claims follow from Lemma 2.16. ∎

2.4 Maps associated to commuting tuple of endomorphisms

2.4.1 Preliminary

Let V be an r-dimensional vector space. Let $\boldsymbol{v} = (v_i)$ be a frame of V. For any endomorphism f of V, we have the matrix $A(f) \in M_r$ determined by $f \cdot \boldsymbol{v} = \boldsymbol{v} \cdot A(f)$, i.e., $f(v_i) = \sum A(f)_{j\,i} \cdot v_j$.

Let f_i $(i=1,2)$ be elements of $End(V)$ such that $f_1\circ f_2=f_2\circ f_1$. We decompose f_i into the product of the unipotent part f_i^u and the semisimple part f_i^s. Then f_1^u, f_1^s, f_2^u and f_2^s are mutually commutative. There exists an appropriate frame $\boldsymbol{v}$ of V such that $A(f_i^s)$ $(i=1,2)$ are diagonal matrices, and $A(f_i^u)$ are upper triangular matrices. Then the matrices $A\bigl(\log f_i^u\bigr)$ are also upper triangular matrices. In the following, we identify the endomorphisms and the matrices via the frame $\boldsymbol{v}$ above.

We put $\eta_i^n:=(2\pi)^{-1}\cdot\log f_i^u$. We also have η_i^s satisfying $\exp\bigl(2\pi\eta_i^s\bigr)=f_i^s$ such that $0\leq \mathrm{Im}(\alpha)<1$ holds for any eigenvalues α of η_i^s. We put $\eta_i:=\eta_i^s+\eta_i^n$, and then we have $\exp\bigl(2\pi\eta_i\bigr)=f_i$.

Let $\varphi:\boldsymbol{R}^2\longrightarrow GL(V)$ be the morphism given by $\varphi(\theta_1,\theta_2):=\exp\bigl(\theta_1\cdot\eta_1+\theta_2\cdot\eta_2\bigr)$. We also put as $\varphi^s(\theta_1,\theta_2):=\exp\bigl(\theta_1\cdot\eta_1^s+\theta_2\cdot\eta_2^s\bigr)$ and $\varphi^u(\theta_1,\theta_2):=\exp\bigl(\theta_1\cdot\eta_1^n+\theta_2\cdot\eta_2^n\bigr)$. We have $\varphi=\varphi^s\cdot\varphi^u$. Under the identification $GL(V)=GL(r)$ via the frame $\boldsymbol{v}$ above, $\varphi^u(\theta_1,\theta_2)$ are upper triangular whose diagonal entries are 1, and $\varphi^s(\theta_1,\theta_2)$ are diagonal matrices.

2.4.2 Construction

Let us take an element $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_r)\in\boldsymbol{R}^r$ such that $\alpha_i>\alpha_{i+1}$ for any i. We put $\beta:=\min\bigl\{\alpha_i-\alpha_{i+1}\bigr\}>0$. We have the C^∞-map $F:\boldsymbol{R}\times\boldsymbol{R}^2\longrightarrow\mathcal{PH}(r)$ given as follows:

$$F(t,\theta_1,\theta_2):=\kappa\Bigl(\varphi(\theta_1,\theta_2),\gamma_{\boldsymbol{\alpha}}(t)\Bigr).$$

We put $\widetilde{\varphi}(t,\theta_1,\theta_2)=\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot\varphi(\theta_1,\theta_2)\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2}$. Similarly we obtain $\widetilde{\varphi}^u$ and $\widetilde{\varphi}^s$. We have $\widetilde{\varphi}^s=\varphi^s$. We also have the following:

$$\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot F(t,\theta_1,\theta_2)\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2}=\widetilde{\varphi}(t,\theta_1,\theta_2)\cdot{}^t\overline{\widetilde{\varphi}(t,\theta_1,\theta_2)}.$$

Lemma 2.18 *For any C_0, there exists a positive number C_1 such that the following inequalities hold, in the case $|\theta_1|+|\theta_2|<C_0$:*

$$\bigl|\bigl|\widetilde{\varphi}^u(t,\theta_1,\theta_2)-I_r\bigr|\bigr|\leq C_1\cdot e^{-\beta\cdot t}.$$

$$\bigl|\bigl|\widetilde{\varphi}(t,\theta_1,\theta_2)-\varphi^s(\theta_1,\theta_2)\bigr|\bigr|\leq C_1\cdot e^{-\beta\cdot t}.$$

$$d_{\mathcal{PH}(r)}\Bigl(\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot F(t,\theta_1,\theta_2)\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2},\ \varphi^s(\theta_1,\theta_2)\cdot{}^t\overline{\varphi^s(\theta_1,\theta_2)}\Bigr)\leq C_1\cdot e^{-\beta\cdot t}.$$

Proof Since $\varphi^u(\theta_1,\theta_2)$ are upper triangular whose diagonal entries are 1, it is easy to check. ∎

2.4.3 Estimate of derivatives

We see the estimate of the derivative of F. We have $\frac{\partial F}{\partial\theta_i}=\eta_i\cdot F+F\cdot{}^t\overline{\eta_i}$.

Lemma 2.19 *For any positive number C_0, there exists a positive number C_1 such that the following holds in the case $|\theta_1|+|\theta_2|<C_0$:*

$$\left|\left|\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot\frac{\partial F}{\partial\theta_i}\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2}-\varphi^s(\theta_1,\theta_2)\cdot\bigl(2\,\mathrm{Re}\,\eta_i^s\bigr)\cdot{}^t\overline{\varphi^s(\theta_1,\theta_2)}\right|\right|\leq C_1\cdot e^{-\beta\cdot t}.$$

As a result, we have the following:

$$\left|\frac{\partial F}{\partial\theta_i}\right|=2\,\bigl|\mathrm{Re}\,\eta_i^s\bigr|+O\bigl(e^{-\beta t}\bigr)=\frac{\rho(f_i)}{2\pi}+O\bigl(e^{-\beta t}\bigr).$$

Proof We have the following convergence when $t\to\infty$:

$$\begin{aligned}
&\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot\frac{\partial F}{\partial\theta_i}\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2}\\
&=\bigl(\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot\eta_i\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2}\bigr)\cdot\bigl(\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot F\cdot\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\bigr)+\bigl(\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot F\cdot\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\bigr)\cdot{}^t\overline{\bigl(\gamma_{\boldsymbol{\alpha}}(t)^{-1/2}\cdot\eta_i\cdot\gamma_{\boldsymbol{\alpha}}(t)^{1/2}\bigr)}\\
&\qquad\longrightarrow\eta_i^s\cdot\varphi^s(\theta_1,\theta_2)^t\overline{\varphi^s(\theta_1,\theta_2)}+\varphi^s(\theta_1,\theta_2)^t\overline{\varphi^s(\theta_1,\theta_2)}\cdot{}^t\overline{\eta_i^s}=2\varphi^s(\theta_1,\theta_2)\cdot\mathrm{Re}(\eta_i^s)\cdot{}^t\overline{\varphi^s(\theta_1,\theta_2)}.
\end{aligned}\tag{8}$$

Here the convergence is estimated by $e^{-\beta t}$. Thus we are done. ∎

Lemma 2.20 *We have the following:*

$$\left|\frac{\partial F}{\partial t}\right|^2 = \left|\frac{d\gamma_{\boldsymbol{\alpha}}(t)}{dt}\right|^2 = \sum \alpha_i^2.$$

Proof It follows from the $GL(r)$-invariance of the metric of $\mathcal{PH}(r)$. ∎

2.4.4 Extension

We use the real coordinate $\zeta_i = \xi_i + \sqrt{-1}\eta_i$ $(i = 1, 2)$ for $\mathbb{H}^2$. Let A, B, and C_i $(i = 1, 2)$ be positive numbers. Let us consider the morphism $\Phi : \mathbb{H}^2 \longrightarrow \mathcal{PH}(r)$ given as follows:

$$\Phi(\zeta_1, \zeta_2) := \gamma_{\boldsymbol{\alpha}}\big(B \cdot \log C_1\eta_1\big)^{-1/2} \cdot F\big(B \cdot \log(C_1\eta_1 + C_2\eta_2 + A), \xi_1, \xi_2\big) \cdot \gamma_{\boldsymbol{\alpha}}\big(B \cdot \log C_1\eta_1\big). \tag{9}$$

We also put as follows:

$$\varphi^\star(\eta_1, \xi_1, \xi_2) := \tilde{\varphi}\big(B \cdot \log C_1\eta_1, \xi_1, \xi_2\big).$$

Then we have the following estimate, due to Lemma 2.18:

$$\big|\big|\varphi^\star(\eta_1, \xi_1, \xi_2) - \varphi^s(\xi_1, \xi_2)\big|\big| = O\big(\eta_1^{-\beta \cdot B}\big).$$

Lemma 2.21 *We have the following:*

$$\lim_{\eta_1 \to \infty} \Phi(\zeta_1, \zeta_2) = \varphi^s(\xi_1, \xi_2) \cdot {}^t\overline{\varphi^s(\xi_1, \xi_2)}.$$

Proof We have the following:

$$\Phi(\zeta_1, \zeta_2) = \varphi^\star(\eta_1, \xi_1, \xi_2) \cdot \gamma_{\boldsymbol{\alpha}}\big(B \cdot \log\big(C_1\eta_1 + C_2\eta_2 + A\big) - B \cdot \log\big(C_1\eta_1\big)\big) \cdot {}^t\overline{\varphi^\star(\eta_1, \xi_1, \xi_2)}.$$

Then it is easy to check the claim. ∎

We take the isomorphism $\boldsymbol{R}_{\geq 0} \longrightarrow [0, 1[$ given by $\eta_1 \longmapsto \kappa = (1 + \eta_1)^{-1} \cdot \eta_1$. It induces the isomorphism $\mathbb{H}^2 \simeq \big(\boldsymbol{R} \times [0, 1[\big) \times \mathbb{H}$, and hence we obtain the map $\tilde{\Phi} : \big(\boldsymbol{R} \times [0, 1[\big) \times \mathbb{H} \longrightarrow \mathcal{PH}(r)$.

Lemma 2.22 *The map $\tilde{\Phi}$ can be naturally extended to the continuous map $\boldsymbol{R} \times [0, 1] \times \mathbb{H} \longrightarrow \mathcal{PH}(r)$. We denote the extended map by $\tilde{\Phi}$. On $\boldsymbol{R} \times \{1\} \times \mathbb{H}$, we have $\tilde{\Phi}(\xi_1, 1, \zeta_2) = \varphi^s(\xi_1, \xi_2) \cdot {}^t\overline{\varphi^s(\xi_1, \xi_2)}$.*

Proof It follows from Lemma 2.21. ∎

We have the map $\tilde{f}_2(\kappa) : [0, 1[\longrightarrow GL(r)$ given as follows:

$$\tilde{f}_2(\kappa) = \gamma_{\boldsymbol{\alpha}}\Big(B \cdot \log \frac{c_1\kappa}{1 - \kappa}\Big)^{-1/2} \cdot f_2 \cdot \gamma_{\boldsymbol{\alpha}}\Big(B \cdot \log \frac{c_1\kappa}{1 - \kappa}\Big)^{1/2}.$$

It is naturally extended to the continuous map $[0, 1] \longrightarrow GL(r)$, which we denote also by $\tilde{f}_2(\kappa)$. We have $\tilde{f}_2(1) = f_2^s$.

Let $\tilde{\Phi}_{(\xi_1, \kappa)}$ denote the restriction of $\tilde{\Phi}$ to $\{(\xi_1, \kappa)\} \times \mathbb{H}$. We have the $\mathbb{Z}$-action on $\mathbb{H}$ by $n \cdot (\xi_2, \eta_2) \longrightarrow (\xi_2 + n, \eta_2)$. For $(\xi_1, \kappa) \in \boldsymbol{R} \times [0, 1]$, the action of $\mathbb{Z}$ on $\mathcal{PH}(r)$ is given by $\tilde{f}_2(\kappa)$. Then we can regard $\tilde{\Phi}_{(\xi_1, \kappa)}$ as the twisted map with respect to the actions, i.e., $\tilde{\Phi}_{(\xi_1, \kappa)} : \overline{\Delta}^* \longrightarrow \mathcal{PH}(r)/\langle \tilde{f}_2(\kappa) \rangle$. (See the subsubsection 2.5.2 for twisted maps.) Thus we obtain the continuous family of C^∞-twisted maps $\big\{\tilde{\Phi}_{(\xi_1, \kappa)} \,\big|\, (\xi_1, \kappa) \in \boldsymbol{R} \times [0, 1]\big\}$.

Lemma 2.23 *We have the estimate, which is independent of (ξ_1, κ):*

$$\left|\frac{\partial \tilde{\Phi}_{(\xi_1, \kappa)}}{\partial \xi_2}\right| = \frac{\rho(f_2)^2}{4\pi^2} + O\Big(\big(C_1\eta_1 + C_2\eta_2 + A\big)^{-B \cdot \beta}\Big),$$

$$\left|\frac{\partial \tilde{\Phi}_{(\xi_1, \kappa)}}{\partial \eta_2}\right| = O\Big(\big(C_1\eta_1 + C_2\eta_2 + A\big)^{-1}\Big).$$

Here we put $\eta_1 = \kappa \cdot (1 - \kappa)^{-1}$.

Proof It from Lemma 2.19 and Lemma 2.20. ▮

Let $g_0 := (2\eta_2 + b)^{-2} \cdot \bigl(d\xi_2^2 + d\eta_2^2\bigr)$ be the Poincaré metric of $\overline{\Delta}^*$, where b denotes a positive constant.

Lemma 2.24 *Assume the constant B in (9) is sufficiently large. There exists a positive constant C, which is independent of (ξ_1, κ), such that the following holds:*

$$\int_{\overline{\Delta}^*} \left| e\bigl(\tilde{\Phi}_{(\xi_1,\kappa)}\bigr) - \frac{\rho(f_2)^2}{4\pi^2} \cdot (2\eta_2 + b)^2 \right| \mathrm{dvol}_{g_0} < C. \tag{10}$$

Proof It follows from the previous lemma. ▮

2.5 Preliminary for harmonic maps and harmonic bundles

2.5.1 The energy function is dominated by the energy

Recall the section 8 in [12]. Let (N, g_N) be a Riemannian manifold with non-positive curvature. Let g be a Riemannian metric of $\Delta(R)^n$, and let $f : \Delta(R)^n \longrightarrow N$ be a differentiable map. Let $e(f)$ be the energy function of f with respect to the metrics g and g_N. Assume that $e(f)$ is integrable, and let $E(f)$ denote the energy.

Lemma 2.25 *Let us take a positive number $R' < R$, and then there exists a positive number $C > 0$ such that the following inequality holds for any f and for any point $P \in \Delta(R')^n$:*

$$e(f)(P) \leq C \cdot E(f).$$

Proof We only give a remark. In [12], it is discussed for the map $f : (M, g_M) \longrightarrow (N, g_N)$ for a compact Riemannian manifold (M, g_M). Since the argument is local, it can be applied in our case. ▮

2.5.2 Twisted map and twisted harmonic map

Recall the correspondence of harmonic metrics and twisted harmonic maps. Let X be a C^∞-manifold and let (E, ∇) be a flat connection on X. Let h be a continuous hermitian metric of E. Let $\pi : \tilde{X} \longrightarrow X$ be a universal covering. Then we obtain the pull backs $\pi^{-1}(E, \nabla, h)$. Once we take a flat frame $\boldsymbol{v}$ of $\pi^{-1}(E, \nabla)$, then the metric $\pi^{-1}h$ induces the C^∞-map $\tilde{\Psi}_h : \tilde{X} \longrightarrow \mathcal{PH}(r)$. If h is C^∞, then $\tilde{\Psi}_h$ is also C^∞.

We have the $\pi_1(X)$-action on $\tilde{X}$. The monodromy induces endomorphism $\rho : \pi_1(X) \longrightarrow GL(r)$, which induces the $\pi_1(X)$-action on $\mathcal{PH}(r)$. Since the map $\tilde{\Psi}_h$ is equivariant, we obtain the continuous map $\Psi_h : X \longrightarrow \mathcal{PH}(r)/\rho\bigl(\pi_1(X)\bigr)$. Although $\mathcal{PH}(r)/\rho\bigl(\pi_1(X)\bigr)$ is not a manifold, we say Ψ_h is C^∞ (resp. harmonic) if $\tilde{\Psi}_h$ is C^∞ (resp. harmonic). We do not explicitly distinguish Ψ_h and $\tilde{\Psi}_h$.

2.5.3 The induced one form

We recall some formalism following [42]. Let X be a complex manifold with Kahler metric g. Let V be a C^∞-vector bundle on X, and ∇ be a flat connection of V. We have the decomposition $\nabla = d' + d''$ into the sum of the $(1,0)$-part and the $(0,1)$-part.

Let K be a hermitian metric of V. Then the differential operators δ' and δ'' is given by the conditions that $\delta' + d''$ and $d' + \delta''$ are unitary connections. We put as follows:

$$\partial = \frac{d' + \delta'}{2}, \quad \overline{\partial} = \frac{d'' + \delta''}{2}, \quad \theta = \frac{d' - \delta'}{2}, \quad \theta^\dagger = \frac{d'' - \delta''}{2}.$$

Lemma 2.26 *The operator $\partial + \overline{\partial}$ is a unitary connection. The one forms θ and $\theta^\dagger$ are mutually adjoint, and hence $\theta + \theta^\dagger$ is self adjoint.* ▮

Lemma 2.27 *We have the following vanishings:*

$$(\partial + \overline{\partial})(\theta + \theta^\dagger) = 0, \qquad (\partial + \overline{\partial})^2 + (\theta + \theta^\dagger)^2 = 0. \tag{11}$$

The first equality means the vanishing of the commutator $[\partial + \overline{\partial},\ \theta + \theta^\dagger]$ of the operators.

Proof We decompose the equality $\nabla^2 = 0$ into the self adjoint part and the anti-self adjoint part. Then we obtain the equalities. ∎

Corollary 2.4 *We have the following vanishings:*

$$\partial\theta = 0, \quad \overline{\partial}\theta^\dagger = 0, \quad \partial\theta^\dagger + \overline{\partial}\theta = 0,$$

$$\partial^2 + \theta^2 = 0, \quad \overline{\partial}^2 + \theta^{\dagger\,2} = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial + [\theta, \theta^\dagger] = 0.$$

Proof We have only to decompose the equalities (11) into $(2,0)$-parts, $(1,1)$-parts and $(0,2)$-parts. ∎

Remark 2.2 *If we have $\theta^2 = 0$ and $\overline{\partial}\theta = 0$, we also have the other vanishings $(\theta^\dagger)^2 = \overline{\partial}^2 = \partial^2 = \partial\theta^\dagger = 0$. In other words, the metric K is pluri-harmonic.* ∎

2.5.4 The description by matrices

Let $\boldsymbol{v}$ be a flat frame of V. Let $H = (h_{i\,j})$ be a $\mathcal{PH}(r)$-valued function determined by $H = H(K, \boldsymbol{v})$. We also have the $M(r)$-valued $(1,0)$-form Θ and the $M(r)$-valued $(0,1)$-form $\Theta^\dagger$ determined by $\theta\boldsymbol{v} = \boldsymbol{v}\cdot\Theta$ and $\theta^\dagger\boldsymbol{v} = \boldsymbol{v}\cdot\Theta^\dagger$. We have the following:

$$\begin{aligned}\overline{\partial}h_{i\,j} &= K(d''v_i, v_j) + K(v_i, \delta' v_j) = K\big(v_i, (\delta' - d')v_j\big) \\ &= -K(v_i, 2\theta v_j) = -2\sum_k K(v_i, \Theta_{k\,j}v_k) = -2\sum_k h_{i\,k}\cdot\overline{\Theta}_{k\,j}. \end{aligned} \tag{12}$$

Thus we obtain $\partial\overline{h_{i\,j}} = -2\overline{h_{i\,k}}\cdot\Theta_{k\,j}$. Namely we have the relation $\Theta = -\frac{1}{2}\cdot\bar{H}^{-1}\cdot\partial\overline{H}$. We also have the following:

$$\begin{aligned}\partial h_{i\,j} &= K(d'v_i, v_j) + K(v_i, \delta'' v_j) = K\big(v_i, (\delta'' - d'')v_j\big) \\ &= -2K(v_i, \theta^\dagger v_j) = -2\sum_k K(v_i, \Theta^\dagger_{k\,j}v_k) = -2\sum_k h_{i\,k}\overline{\Theta}^\dagger_{k\,j}. \end{aligned} \tag{13}$$

Thus we obtain the relation $\overline{\partial h}_{i\,j} = \sum -2\overline{h}_{i\,k}\cdot\Theta^\dagger_{k\,j}$. Namely we have the relation: $\Theta^\dagger = -\frac{1}{2}\cdot\overline{H}^{-1}\cdot\partial\overline{H}$. Thus we obtain the following relation:

$$\Theta + \Theta^\dagger = -\frac{1}{2}\cdot\overline{H}^{-1}\cdot d\overline{H}.$$

Let Ψ_K denote the twisted map $X \longrightarrow \mathcal{PH}(r)/\pi_1(X)$ associated to K. Let $\boldsymbol{e} = (e_1, \ldots, e_{2n})$ be an orthonormal base of T_PX. We have the metric of $End(E)\otimes\Omega^1_X$ induced by the metrics K and g. By using Lemma 2.3 and Lemma 2.8, we obtain the following:

$$8\cdot||\theta||^2_{K,g} = 4\cdot||\theta+\theta^\dagger||^2_{K,g} = 4\sum_i||\Theta(e_i)+\Theta^\dagger(e_i)||^2_K = \sum_i \mathrm{tr}\Big(\overline{H}^{-1}\cdot d\overline{H}(e_i)\cdot\overline{H}^{-1}\cdot d\overline{H}(e_i)\Big) = e(\Psi_K). \tag{14}$$

2.5.5 Bochner type formula due to Corlette

We put $\vartheta = \theta + \theta^\dagger$. We also put $D^+ = \partial + \overline{\partial}$. Let $\boldsymbol{e} = (e_i)$ be an orthonormal frame of $\bigwedge^{1,0}(T^\vee)$. Then we have the naturally induced orthonormal frame $\bar{\boldsymbol{e}}$ of $\bigwedge^{0,1}(T^\vee)$. Recall that $(1,1)$-form $\sum\varphi_{i\,j}\cdot e_i\wedge\bar{e}_j$ is called primitive if $\sum_i\varphi_{i\,i} = 0$.

Lemma 2.28 (Corlette) *K is harmonic metric if and only if the $(1,1)$-form $\overline{\partial}\theta$ is primitive.*

Proof See the page 376 in [3]. We give only a remark on the difference of the notation. Our ϑ, θ, and ∂ are denoted by θ, $\theta^{1,0}$ and ∂^+ respectively in [3]. ∎

Let ω be the Kahler form of the Kahler manifold (X, g). Let e_i and $\bar{e}_i$ be as above.

Lemma 2.29 *There exists a negative constant C_0 depending only on $\dim X$ such that the following holds, for any $i \neq j$:*

$$e_i \wedge \bar{e}_i \wedge e_j \wedge \bar{e}_j \wedge \omega^{n-2} = C_0 \cdot \omega^n.$$

Proof Recall that ω is described as $2^{-1} \cdot \sqrt{-1} \sum e_i \wedge \bar{e}_i$. Then the claim immediately follows. ∎

Let h be a harmonic metric of a flat bundle (E, ∇). Then we have the induced hermitian metric $\langle \cdot, \cdot \rangle$ of $End(E)$. It naturally induces the following pairing, which we also denote by $\langle \cdot, \cdot \rangle$:

$$\big(\mathrm{End}(E) \otimes \Omega^{\cdot\cdot}_X\big) \otimes \overline{\big(\mathrm{End}(E) \otimes \Omega^{\cdot\cdot}_X\big)} \longrightarrow \Omega^{\cdot\cdot}_X.$$

For example, we have the following:

$$\big\langle f \cdot dz^I \wedge d\bar{z}^J,\, g \cdot dz^K \wedge d\bar{z}^L \big\rangle = \langle f, g \rangle \cdot dz^I \wedge d\bar{z}^J \wedge d\bar{z}^K \wedge dz^L.$$

Lemma 2.30 *We have the following formula:*

$$\partial\overline{\partial}\big\langle \theta,\, \theta \big\rangle = -\frac{1}{2}\big\langle [\theta,\theta], [\theta,\theta] \big\rangle + \big\langle \overline{\partial}\theta, \overline{\partial}\theta \big\rangle. \tag{15}$$

Proof We have the following:

$$\partial\overline{\partial}\big\langle \theta, \theta \big\rangle = \partial\big\langle \overline{\partial}\theta, \theta \big\rangle = \big\langle \partial\overline{\partial}\theta, \theta \big\rangle + \big\langle \overline{\partial}\theta, \overline{\partial}\theta \big\rangle.$$

Due to the vanishings $\partial\overline{\partial} + \overline{\partial}\partial + [\theta, \theta^\dagger] = 0$ and $\partial\theta = 0$, we have the following:

$$\partial\overline{\partial}\theta = -\overline{\partial}\partial\theta - \big[[\theta, \theta^\dagger], \theta\big] = \frac{1}{2}\big[[\theta,\theta], \theta^\dagger\big] = -\frac{1}{2}\big[\theta^\dagger, [\theta,\theta]\big].$$

Since we have $\big\langle \big[\theta^\dagger, [\theta,\theta]\big],\, \theta \big\rangle = \big\langle [\theta,\theta],\, [\theta,\theta] \big\rangle$, we obtain (15). ∎

Thus we obtain the equality $\partial\overline{\partial}\big\langle \theta,\, \theta \big\rangle \cdot \omega^{n-2} = -\frac{1}{2}\big\langle [\theta,\theta],\, [\theta,\theta] \big\rangle \cdot \omega^{n-2} + \big\langle \overline{\partial}\theta,\, \overline{\partial}\theta \big\rangle \cdot \omega^{n-2}$.

We have the description $\theta = \sum f_i \cdot e_i$ for $f_i \in \mathrm{End}(E)$. Then we have the following:

$$[\theta, \theta] = 2 \sum_{i<j} [f_i, f_j] \cdot e_i \wedge e_j.$$

Lemma 2.31 *We have the following formula:*

$$-\frac{1}{2}\big\langle [\theta,\theta],\, [\theta,\theta] \big\rangle \cdot \omega^{n-2} = \sum_{i,j} \big|[f_i, f_j]\big|_h^2 \cdot C_0 \cdot \omega^n. \tag{16}$$

Proof In the case $i < j$ and $k < l$, we have the following equality:

$$\big\langle [f_i, f_j],\, [f_k, f_l] \big\rangle \cdot e_i \wedge e_j \wedge \bar{e}_k \wedge \bar{e}_l \wedge \omega^{n-2} = \begin{cases} -\big|[f_i, f_j]\big|_h^2 \cdot C_0 \cdot \omega^n & (i = k, j = l, i \neq j), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we immediately obtain (16). ∎

We have the description $\overline{\partial}\theta = \sum \varphi_{i,j} \cdot e_i \wedge \bar{e}_j$.

Lemma 2.32 *We have the following formula:*

$$\big\langle \overline{\partial}\theta,\, \overline{\partial}\theta \big\rangle \cdot \omega^{n-2} = C_0 \cdot \left(\sum_{i,j} \big\langle \varphi_{i,j}, \varphi_{i,j} \big\rangle \right) \cdot \omega^n. \tag{17}$$

Proof We have the following:

$$\begin{aligned}\big\langle\overline{\partial}\theta,\,\overline{\partial}\theta\big\rangle\cdot\omega^{n-2} &= \sum_{i,j,k,l}\big\langle\varphi_{i\,j},\,\varphi_{k,l}\big\rangle\cdot e_i\wedge\bar{e}_j\wedge\bar{e}_k\wedge e_l\wedge\omega^{n-2}\\ &=\sum_{i\neq j}\big\langle\varphi_{i,j},\,\varphi_{i,j}\big\rangle e_i\wedge\bar{e}_i\wedge e_j\wedge\bar{e}_j\wedge\omega^{n-2}-\sum_{i\neq k}\big\langle\varphi_{i,i},\,\varphi_{k,k}\big\rangle e_i\wedge\bar{e}_i\wedge e_k\wedge\bar{e}_k\wedge\omega^{n-2}\\ &=C_0\cdot\Big(\sum_{i\neq j}\big\langle\varphi_{i,j},\,\varphi_{i,j}\big\rangle-\sum_{i\neq k}\big\langle\varphi_{i,i},\varphi_{k,k}\big\rangle\Big)\cdot\omega^n. \end{aligned}\tag{18}$$

Recall we have $\sum_i\varphi_{i,i}=0$, for $\overline{\partial}\theta$ is primitive. Therefore we have the following equality:

$$\sum_i\langle\varphi_{i,i},\varphi_{i,i}\rangle+\sum_{i\neq k}\langle\varphi_{i,i},\varphi_{k,k}\rangle=0.$$

Then we immediately obtain (17). ∎

Proposition 2.1 *There exist negative constants C_1 and C_2 depending only on $\dim X$ such that the following holds:*

$$\partial\overline{\partial}\big\langle\theta,\,\theta\big\rangle\cdot\omega^{n-2}=\Big(C_1\cdot\big|[\theta,\theta]\big|_h^2+C_2\cdot\big|\overline{\partial}\theta\big|_h^2\Big)\cdot\omega^n.\tag{19}$$

Proof It immediately follows from Lemma 2.31 and Lemma 2.32. ∎

The formula (19) is the Bochner type formula due to Corlette. Recall the argument to derive the pluri-harmonicity of the harmonic metric h, when X is compact Kahler. If X is compact, we have the vanishing:

$$0=\int_X\partial\overline{\partial}\big\langle\theta,\theta\big\rangle\cdot\omega^{n-2}=C_1\int_X\big|[\theta,\theta]\big|^2+C_2\int_X\big|\overline{\partial}\theta\big|^2.$$

It implies $[\theta,\theta]=\overline{\partial}\theta=0$ on X, which means the metric h is pluri-harmonic.

2.5.6 A variation of the Bochner type formula

Let f and g be sections of $\mathrm{End}(E)$. We denote the adjoint of them by $f^\dagger$ and $g^\dagger$ respectively.

Lemma 2.33 *We have the equality:* $\big\langle f\cdot dz_i,\ g\cdot dz_j\big\rangle=-\big\langle g^\dagger\cdot d\bar{z}_j,\ f^\dagger\cdot d\bar{z}_i\big\rangle$.

Proof It follows from the equality $\langle f,g\rangle=\langle g^\dagger,f^\dagger\rangle$. ∎

Corollary 2.5 *We have the equality* $\partial\overline{\partial}\big\langle\theta,\ \theta\big\rangle=\overline{\partial}\partial\big\langle\theta^\dagger,\ \theta^\dagger\big\rangle$. ∎

For our argument to derive the pluri-harmonicity in quasi projective case, we will also use the following formula.

Proposition 2.2 *We have the following formula:*

$$d\Big(\big\langle\overline{\partial}\theta,\,\theta-\theta^\dagger\big\rangle\wedge\omega^{n-2}\Big)=2\Big(C_1\cdot\big|[\theta,\theta]\big|_h^2+C_2\cdot\big|\overline{\partial}\theta\big|_h^2\Big)\cdot\omega^n.\tag{20}$$

Here the negative constants C_i are same as those in Proposition 2.1.

Proof We have the following:

$$\partial\overline{\partial}\big\langle\theta,\theta\big\rangle\wedge\omega^{n-2}=\partial\big\langle\overline{\partial}\theta,\theta\big\rangle\wedge\omega^{n-2}=\partial\big\langle\overline{\partial}\theta,\theta-\theta^\dagger\big\rangle\wedge\omega^{n-2}.$$

Similarly, we have the equality $\overline{\partial}\partial\langle\theta^\dagger,\theta^\dagger\big\rangle\wedge\omega^{n-2}=\overline{\partial}\big\langle\partial\theta^\dagger,\theta^\dagger-\theta\big\rangle\wedge\omega^{n-2}$, which can be rewritten as follows:

$$\overline{\partial}\big\langle-\overline{\partial}\theta,\ -\big(\theta-\theta^\dagger\big)\big\rangle\wedge\omega^{n-2}=\overline{\partial}\big\langle\overline{\partial}\theta,\theta-\theta^\dagger\big\rangle\wedge\omega^{n-2}.$$

Here we have used Corollary 2.4. Then we obtain the following:

$$\partial\overline{\partial}\langle\theta,\theta\big\rangle\wedge\omega^{n-2}+\overline{\partial}\partial\langle\theta^\dagger,\theta^\dagger\big\rangle\wedge\omega^{n-2}=d\Big(\big\langle\overline{\partial}\theta,\theta-\theta^\dagger\big\rangle\wedge\omega^{n-2}\Big).$$

Then (20) follows from Corollary 2.5 and Proposition 2.1. ∎

3 Tame pure imaginary harmonic bundle

3.1 Definition

Let X be a complex manifold, and $D = \bigcup_i D_i$ be a normal crossing divisor. Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle on $X - D$. Let P be a point of X. Let us take a neighbourhood P with a coordinate $(z_1, \dots, z_n)$ such that $U \cap D = \bigcup_{i=1}^{l} \{z_i = 0\}$. We have the description $\theta = \sum_{i=1}^{l} f_i \cdot dz_i/z_i + \sum_{j=l+1}^{n} g_j \cdot dz_j$. Recall that $(E, \overline{\partial}_E, \theta, h)$ is called tame if the coefficients of the characteristic polynomials $\det(t - f_i)$ and $\det(t - g_j)$ are holomorphic.

Lemma 3.1 *The harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is tame if and only if there exists a holomorphic vector bundle $\tilde{E}$ with a regular Higgs field $\tilde{\theta} \in \mathrm{End}(\tilde{E}) \otimes \Omega_X^{1,0}(\log D)$ such that $(\tilde{E}, \tilde{\theta})_{|X-D} = (E, \theta)$.*

Proof In the subsection 8.6 in [33], it is proved that the prolongment ${}^\diamond E$ by an increasing order is locally free and that θ naturally induces the regular Higgs field on ${}^\diamond E$. On the contrary, it is easy to check $(E, \overline{\partial}_E, \theta, h)$ is tame if there exists a prolongment $(\tilde{E}, \tilde{\theta})$. ∎

Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $X - D$. Let $(\tilde{E}, \tilde{\theta})$ be any prolongment of (E, θ). Let $\mathrm{Res}_i(\tilde{\theta})$ denote the residue of $\tilde{\theta}$ with respect to the irreducible component D_i of D.

Lemma 3.2 *Let P be a point of D_i. The eigenvalues $\mathrm{Res}_i(\tilde{\theta})_{|P}$ is independent of a choice of a prolongment $(\tilde{E}, \tilde{\theta})$.*

Proof The eigenvalues are solutions of $\det(t - f_i)_{|P}$. Since $\det(t - f_i)$ is determined independently of $(\tilde{E}, \tilde{\theta})$, we are done. ∎

Recall that the eigenvalues of $\mathrm{Res}_i(\theta)_{|P}$ is independent of a choice of $P \in D_i$, which is proved in the subsection 8.1 in [33].

Definition 3.1 *If any eigenvalues of the residues $\mathrm{Res}_i(\tilde{\theta})$ is pure imaginary, $(E, \overline{\partial}_E, \theta, h)$ is called a tame pure imaginary harmonic bundle.* ∎

3.2 Tame pure imaginary harmonc bundle on a punctured disc

3.2.1 The estimate of Higgs field

We use the Poincaré metric $g_0 := |z|^{-2} \cdot (-\log|z|^2 + A)^{-2} \cdot dz \cdot d\bar{z}$ on $\overline{\Delta}^*$. Here A denotes a positive number. Let us consider a tame harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on a punctured disc Δ^*. We have the prolongment ${}^\diamond E$ and the description $\theta = f_0 \cdot dz/z$ for some $f_0 \in \mathrm{End}({}^\diamond E)$ on Δ. We put $\mathcal{S}p(\theta) := \mathcal{S}p(f_{0\,|\,O})$. For any $\alpha \in \mathcal{S}p(\theta)$, we denote $\dim \mathbb{E}({}^\diamond E, \alpha)$ by $\mathfrak{m}(\alpha)$. We put as follows:

$$\mathfrak{t}(\theta) = \sum_{\alpha \in \mathcal{S}p(\theta)} \mathfrak{m}(\alpha) \cdot |\alpha|^2.$$

The following lemma can be elementarily shown.

Lemma 3.3 *There exist positive constants R, C and ϵ, depending on $(E, \overline{\partial}_E, \theta)$, such that the following holds:*

- *For any point $P \in \Delta^*(R)$ and for any eigenvalue of $f_{0\,|\,P}$, there exists the unique element $\alpha \in \mathcal{S}p(\theta)$ such that the following holds:*
$$|a - \alpha| \le C \cdot |z(P)|^{\epsilon}.$$

∎

Lemma 3.4 *There exist positive constants R', C', ϵ', depending only on R, C, ϵ in Lemma 3.3, such that the following claims hold.*

- *We have the decomposition $E=\bigoplus_{\alpha\in\mathcal{S}p(\theta)}E_\alpha$ on $\Delta^*(R')$. The decomposition is preserved by f_0. For any point $P\in\Delta^*(R')$, and for any $v\in E_{\alpha_i\,|\,P}$ and $w\in E_{\alpha_j\,|\,P}$ $(\alpha_i\neq\alpha_j)$, the following inequality holds:*
$$h(v,w)\leq C'\cdot|z(P)|^{\epsilon'}\cdot|v|_h\cdot|w|_h.$$
- *The following inequality holds:*
$$\Big||f_0|_h^2-\mathfrak{t}(\theta)\Big|\leq C'\cdot(-\log|z|^2+A)^{-2}. \tag{21}$$

Proof The claims are proved in the subsection 7.1 of [33]. ∎

Corollary 3.1 *There exists constant C_0, depending only on R,C,ϵ in Lemma 3.3, such that the following holds:*
$$\Big||\theta|_{h,g_0}^2-2\mathfrak{t}(\theta)\cdot\big(-\log|z|^2\big)^2\Big|\leq C_0. \tag{22}$$

Proof It immediately follows from Lemma 3.4. ∎

Lemma 3.3 and Lemma 3.4 hold for any tame harmonic bundle on Δ^*. In the pure imaginary case, the estimate (22) can be refined. We put $\vartheta=\theta+\theta^\dagger$. We use the polar coordinate $z=r\cdot\exp(\sqrt{-1}\eta)$. We denote $\partial/\partial r$ and $\partial/\partial\eta$ by ∂_r and ∂_η respectively.

Lemma 3.5 *Assume that $(E,\overline{\partial}_E,\theta,h)$ is pure imaginary. There exists a positive number C'', depending only on R,C,ϵ in Lemma 3.3, such that the following inequalities hold:*
$$\Big|\big|\vartheta\big(\partial_\eta\big)\big|_h^2-4\mathfrak{t}(\theta)\Big|\leq C''\cdot\big(-\log r^2+A\big)^{-2}.$$
$$\big|\vartheta\big(\partial_r\big)\big|_h^2\leq C''\cdot r^{-2}\cdot\big(-\log r^2+A\big)^{-2}.$$

Proof We have the following description:
$$\vartheta=\big(f+f^\dagger\big)\frac{dr}{r}+\sqrt{-1}\big(f-f^\dagger\big)d\eta.$$
Then the claims follow from Lemma 3.4. ∎

3.2.2 The estimate of the energy of the associated twisted harmonic map

Let $(E,\overline{\partial}_E,\theta,h)$ be a tame pure imaginary harmonic bundle. Let $(\mathcal{E}^1,\mathbb{D}^1)$ be the flat bundle associated to $(E,\overline{\partial}_E,\theta,h)$ on $\overline{\Delta}^*$.

Lemma 3.6 *The KMS-spectrum of $\mathcal{E}^1$ is given as follows:*
$$\mathcal{KMS}(\mathcal{E}^1)=\big\{(b,2\sqrt{-1}c-b)\,\big|\,(b,\sqrt{-1}c)\in\mathcal{KMS}(\mathcal{E}^0)\big\},$$
$$\mathcal{KMS}^f(\mathcal{E}^1)=\big\{\big(0,\exp(2\pi\sqrt{-1}b+4\pi c)\big)\,\big|\,(b,\sqrt{-1}c)\in\overline{\mathcal{KMS}}(\mathcal{E}^0)\big\}.$$

Proof See the the section 5 in [39] or the subsection 7.3 and 7.4 in [33]. We only remark the following equalities:
$$\mathfrak{p}(1,b,\sqrt{-1}c)=b,\quad \mathfrak{e}(1,b,\sqrt{-1}c)=2\sqrt{-1}c-b,$$
$$\mathfrak{p}^f(1,b,\sqrt{-1}c)=\mathrm{Re}\big(1\cdot\sqrt{-1}c+1\cdot\overline{\sqrt{-1}c}\big)=0,\qquad \mathfrak{e}^f(1,b,\sqrt{-1}c)=\exp\big(2\pi\sqrt{-1}b+4\pi c\big).$$
∎

Let φ be the monodromy of the flat bundle $(\mathcal{E}^1,\mathbb{D}^1)$.

Lemma 3.7 *We have $\rho(\varphi)^2=64\pi^2\cdot\mathfrak{t}(\theta)$. (See the subsubsection 2.3.3 for ρ.)*

Proof From Lemma 3.6, we obtain the following:

$$\rho(\varphi)^2 = \sum_{(b,\sqrt{-1}c)\in\mathcal{KMS}(\mathcal{E}^0)} \mathfrak{m}(b,\sqrt{-1}c)\cdot(2\cdot 4\pi c_i)^2 = 64\pi^2 \sum_{\sqrt{-c}\in\mathcal{S}p(\theta)} \mathfrak{m}(\sqrt{-1}c)\cdot c^2 = 64\pi^2\cdot\mathfrak{t}(\theta).$$

Thus we are done. ∎

Let Ψ_h be the twisted harmonic map $\Delta^* \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$ associated with $(\mathcal{E}^1,\mathbb{D}^1,h)$. Recall that we have $e(\Psi_h) = 8\cdot|\theta|^2_{h,g_0}$ due to (14).

Lemma 3.8 *There exists a positive constant C_1, depending only on R, C, ϵ in Lemma 3.3, such that the following holds:*

$$\Big|e(\Psi_h) - \frac{\rho(\varphi)^2}{4\pi^2}\cdot\big(-\log|z|^2 + A\big)^2\Big| \le C_1.$$

As a result, we obtain the following finiteness:

$$\int_{\Delta^*}\Big|e(\Psi_h) - \frac{\rho(\varphi)^2}{4\pi^2}\cdot\big(-\log|z|^2 + A\big)^2\Big|\cdot \mathrm{dvol}_{g_0} < \infty.$$

Proof It immediately follows from $e(\Psi_h) = 8\cdot|\theta|^2_{h,g_0}$ and the estimate of θ. ∎

3.2.3 A characterization of tame pure imaginary harmonic bundle on a punctured disc

We put $T(R_1,R_2) := \big\{z\in \boldsymbol{C}\,\big|\, R_1 \le -\log|z| \le R_2\big\}$, and $T(R) := T(0,R)$. We use the Poincaré metric $g := |z|^{-2}\cdot\big(-\log|z|^2 + A\big)^{-2} dz\cdot d\bar{z}$ on $\overline{\Delta}^*$. We use the real coordinate $z = \exp\big(\sqrt{-1}x - y\big)$.

Lemma 3.9 *Let (E,∇) be a flat bundle on $T(R_1,R_2)$ with the monodromy φ. Let h be a hermitian metric of (E,∇), and let $\Psi_h : T(R_1,R_2) \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$ be the corresponding twisted map. Then we have the following a priori lower bound of the energy:*

$$\int_{T(R_1,R_2)} e\big(\Psi_h\big)\,\mathrm{dvol}_g \ge \int_{T(R_1,R_2)} \big|\partial_x\Psi_h\big|^2\cdot\big(2y+A\big)^2\,\mathrm{dvol}_g \ge \int_{T(R_1,R_2)} \frac{\rho(\varphi)^2}{4\pi^2}\big(2y+A\big)^2\cdot\mathrm{dvol}_g = \frac{\rho(\varphi)^2}{2\pi}(R_2 - R_1). \tag{23}$$

Proof We always have the inequality $e(\Psi_h) \ge \big|\partial_x\Psi_h\big|^2\cdot\big|\partial_x\big|^{-2} = \big|\partial_x\Psi_h\big|^2\cdot(2y+A)^2$. We have the following inequality, due to Lemma 2.15:

$$\int_0^{2\pi}\left|\frac{\partial\Psi_h}{\partial x}\right|^2\cdot dx \ge \frac{\rho(\varphi)^2}{2\pi}.$$

Then the inequalities (23) immediately follows. ∎

Proposition 3.1 *Let $(E,\overline{\partial}_E,\theta,h)$ be a harmonic bundle on a punctured disc $\overline{\Delta}^*$. Let φ be the monodromy of the corresponding flat bundle $\big(\mathcal{E}^1,\mathbb{D}^1\big)$, and let $\Psi_h : \overline{\Delta}^* \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$ denote the corresponding twisted harmonic map. Assume that there exists an integrable function F on $\overline{\Delta}^*$ with respect to the measure* dvol_g *satisfying the following, for any sufficiently large R:*

$$\int_{T(R)} e\big(\Psi_h\big)\,\mathrm{dvol}_g \le \int_{T(R)}\left(\frac{\rho(\varphi)^2}{4\pi^2}(2y+A)^2 + F\right)\mathrm{dvol}_g\,. \tag{24}$$

Then $(E,\overline{\partial}_E,\theta,h)$ is tame and pure imaginary.

Proof First we see that the harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ is tame. Due to Lemma 3.9, we have the lower bound of the energy for any R:

$$\int_{T(R)} e\big(\Psi_h\big)\cdot\mathrm{dvol}_g \ge \int_{T(R)}\frac{\rho(\varphi)^2}{4\pi^2}\cdot\big(2y+A\big)^2\cdot\mathrm{dvol}_g\,. \tag{25}$$

From (24) and (25), we obtain the following inequality:

$$\int_{T(R_1,R_2)} e\big(\Psi_h\big)\cdot \mathrm{dvol}_g \le \int_{T(R_1,R_2)} \frac{\rho(\varphi)^2}{4\pi^2}\cdot\big(2y+A\big)^2\cdot \mathrm{dvol}_g + \int_{T(0,R_2)} F\cdot \mathrm{dvol}_g \le \frac{\rho(\varphi)^2}{2\pi}\cdot(R_2-R_1)+C. \quad (26)$$

Here we put $C=\int_{\overline{\Delta}^*} F\cdot \mathrm{dvol}$.

Let us consider the universal covering $\mathbb{H}\longrightarrow\Delta^*$ given by $x+\sqrt{-1}y\longmapsto z=\exp\big(\sqrt{-1}x-y\big)$. We have the induced Poincaré metric $g=(2y+A)^{-2}\cdot\big(dx\cdot dx+dy\cdot dy\big)$ on $\mathbb{H}$. We have the induced map $\Psi_h:\mathbb{H}\longrightarrow\mathcal{PH}(r)$. We put as follows:

$$\tilde{S}(x_0,y_0):=\big\{x+\sqrt{-1}y\,\big|\,y_0-1\le y\le y_0+1,\ x_0-\pi\le x\le x_0+\pi\big\}.$$

From (26), we have the following inequality, for the energy of Ψ_h:

$$\int_{\tilde{S}(x_0,y_0)} e(\Psi_h)\cdot \mathrm{dvol}_g \le \frac{\rho(\varphi)^2}{\pi}+C. \quad (27)$$

Let $g_1:=dx\cdot dx+dy\cdot dy$ be the Euclidean metric. Let $e_{g_1}(\Psi_h)$ denote the energy function of Ψ_h with respect to the metric g_1. From (27), we have the following inequality:

$$\int_{\tilde{S}(x_0,y_0)} e_{g_1}(\Psi_h)\cdot \mathrm{dvol}_{g_1} \le \frac{\rho(\varphi)^2}{\pi}+C. \quad (28)$$

Lemma 3.10 *We have the estimate* $e(\Psi_h)=O\big((2y+A)^2\big)$.

Proof Since the right hand side of (28) is independent of a choice of (x_0,y_0), we obtain the constant C_1 such that $e_{g_1}\big(\Psi_h\big)\le C_1$ on $\mathbb{H}$, due to Lemma 2.25. Since we have the relation $e(\Psi_h)=e_{g_1}(\Psi_h)\cdot\big(2y+A\big)^2$, we obtain the estimate $e\big(\Psi_h\big)\le C_1\cdot(2y+A)^2$. ∎

Recall that we have the relation $8\cdot\big|\theta\big|^2=e\big(\Psi_h\big)$. Let us describe $\theta=f\cdot dz/z$, and then we have $\big|\theta\big|^2=2\cdot|f|_h^2\cdot(2y+A)^2$. Hence we obtain the boundedness of $|f|_h$ on $\overline{\Delta}^*$. Then it is easy to derive that the coefficients of $\det(t-f)$ are holomorphic on $\overline{\Delta}$, namely, the harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ is tame.

Let us show that the harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ is pure imaginary.

Lemma 3.11 $\big|\partial_y\Psi_h\big|^2\cdot(2y+A)^2$ *is integrable on* $\overline{\Delta}^*$ *with respect to the measure* dvol_g.

Proof From (23), we have the following inequality:

$$\int_{T(R)}\big|\partial_x\Psi_h\big|^2\cdot(2y+A)^2\cdot \mathrm{dvol}_g \ge \int_{T(R)} \frac{\rho(\varphi)^2}{4\pi^2}\cdot\big(2y+A\big)^2\cdot \mathrm{dvol}_g\,. \quad (29)$$

From (24) and (29), we obtain the following inequality for any R:

$$\int_{T(R)}\big|\partial_y\Psi_h\big|^2\cdot(2y+A)^2\cdot \mathrm{dvol}_g \le \int_{T(R)} F\cdot \mathrm{dvol}_g\,.$$

It implies the integrability of $\big|\partial_y\Psi_h\big|^2\cdot(2y+A)^2$. ∎

We have $\theta=f\cdot dz/z=f\cdot\big(\sqrt{-1}dx-dy\big)$. Let $f^\dagger$ denote the adjoint of f, and then we have $\theta^\dagger=f^\dagger\cdot\big(-\sqrt{-1}dx-dy\big)$. We have the following equalities:

$$\big|\partial_y\Psi_h\big|^2\cdot(2y+A)^2=4\cdot\big|\theta(\partial_y)+\theta^\dagger(\partial_y)\big|^2\cdot(2y+A)^2=4\cdot\big|f+f^\dagger\big|^2\cdot(2y+A)^2.$$

Since we have already known that the harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ is tame, we have the following estimate, due to Lemma 3.4:

$$\big|f+f^\dagger\big|^2\cdot(2y+A)^2=\sum_{a\in\mathcal{S}p(\theta)}\big|2\,\mathrm{Re}(a)\big|^2\cdot(2y+A)^2+O\big(1\big). \quad (30)$$

Then we obtain the following vanishing, from the integrability of $\left|\partial_y \Psi_h\right|^2 \cdot (2y+A)^2$ on $\overline{\Delta}^*$ with respect to the measure dvol_g:

$$\sum_{a \in \mathcal{S}p(\theta)} \left|2\,\mathrm{Re}(a)\right|^2 = 0.$$

Namely the harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is pure imaginary. Therefore the proof of Proposition 3.1 is accomplished. ∎

3.3 Semisimplicity

3.3.1 Statement

Proposition 3.2 *Assume that X is a smooth projective variety, and D is a normal crossing divisor of X. Let $(E, \overline{\partial}_E, \theta, h)$ be a tame pure imaginary harmonic bundle on $X - D$. Then the corresponding flat bundle $(\mathcal{E}^1, \mathbb{D}^1)$ is semisimple.* ∎

We will prove the proposition in the next subsubsections 3.3.2–3.3.4. We will also prove the reverse of the proposition in the next sections.

3.3.2 Stability and semistability

Let C be a smooth quasi projective curve over $\boldsymbol{C}$, and $\overline{C}$ be a smooth projective completion. We put $D = \overline{C} - C = \{P_1, \ldots, P_l\}$. Let E be a holomorphic bundle over $\overline{C}$. Let us consider a neighbourhood U of P with a coordinate z such that $z(P_i) = 0$.

Lemma 3.12 *The following data are equivalent:*

- *A filtration F' of $E_{|P_i}$ indexed by $]-1, 0]$.*
- *A filtration F of $\bigcup_h E(h \cdot P_i)$ indexed by $\boldsymbol{R}$ such that $F_0 = E$ and $F_a \cdot z^{-1} = F_{a+1}$.*

Proof Let F' be a filtration of $E_{|P}$ indexed by $]-1, 0]$. We put $F_a := \left\{f \in E \,|\, f_{|P} \in F'_a\right\}$ for any number $a \in]-1, 0]$. For any real number $a \in \boldsymbol{R}$, we take the number $a_0 \in]-1, 0]$ and the integer a_1 which are uniquely determined by $a_0 + a_1 = a$. Then we put $F_a := F_{a_0} \cdot z^{-a_1}$. Thus we obtain the filtration of $\bigcup E(h \cdot P_i)$ satisfying the condition.

The claim in the reverse direction can be shown similarly. ∎

We will not distinguish two kind of data in Lemma 3.12. They are called the parabolic structure of E.

Let F be a filtration of $\bigcup_h E(h \cdot D)$ as above, i.e., we are given filtrations F' of $E_{|P_i}$ $(P_i \in D)$. For any $a \in]-1, 0]$, we put $\mathfrak{m}(a) = \sum_{P_i} \dim \mathrm{Gr}_a^{F'}(E_{|P_i})$. We put as follows:

$$\deg(E, F) := \deg(E) - \sum_{a \in]-1,0]} a \cdot \mathfrak{m}(a), \qquad \mu(E, F) := \frac{\deg(E, F)}{\mathrm{rank}\, E}. \tag{31}$$

Remark 3.1 *The formula* (31) *is same as that given in the section* 6 *of* [39]. *Note our parabolic filtration is increasing.* ∎

A connection of $E_{|C}$ is called regular, if we have $\nabla f \in F_a(E) \otimes \Omega^{1,0}(\log D)$ for any $f \in F_a(E)$.

Let $E' \subset E$ be a subsheaf, then F induces the filtration of E' by $F_a(E') = F_a(E) \cap E'$. We denote it also by F. Recall that the filtered regular connection (E, F, ∇) is called stable, if the inequality $\mu(E', F) < \mu(E, F)$ holds for any sub-connection (E', F, ∇). We recall only a part of the Kobayashi-Hitchin correspondence for harmonic metric (Theorem 5 in [39]):

Proposition 3.3 *Let (E, ∇, h) be a tame harmonic bundle on C. We obtain the filtration F of E by an increasing order. Then the regular filtered connection (E, ∇, F) is polystable.* ∎

3.3.3 Quasi canonical prolongment and the canonical filtration

Let X be a complex manifold, and D be a normal crossing divisor of X. Let (E,∇) be a flat bundle on $X-D$. Then we have the quasi canonical prolongment $QC(E)$ of E:

- $QC(E)$ is a holomorphic vector bundle on X.
- For any $f\in QC(E)$, we have $\nabla(f)\in QC(E)\otimes\Omega^{1,0}(\log D)$.
- Let α be any eigenvalue of $\mathrm{Res}(\nabla)$ for any irreducible component of D. Then it satisfies $0\leq \mathrm{Re}(\alpha)<1$.

For the quasi canonical filtration, we have the canonical filtration of $QC(E)_{|D_i}$ for any irreducible component D_i of D. For simplicity, we only consider the case that $X=\overline{C}$ is a smooth projective curve. We put $C=\overline{C}-D$. Let P_i be a point of D. Then we have the generalized eigen decomposition with respect to the residue $\mathrm{Res}_{P_i}(\nabla)$:

$$QC(E)_{|P_i}=\bigoplus_{\alpha\in\boldsymbol{C}}\mathbb{E}\bigl(\mathrm{Res}_{P_i}(\nabla),\,\alpha\bigr).$$

Here $\mathbb{E}\bigl(\mathrm{Res}_{P_i}(\nabla),\alpha\bigr)$ denotes $\mathrm{Ker}\bigl(\mathrm{Res}_{P_i}(\nabla)-\alpha\bigr)^N$ for any sufficiently large integer N. Then we put as follows, for any $-1<a\leq 0$:

$$F_a\bigl(QC(E)_{|P_i}\bigr)=\bigoplus_{-\mathrm{Re}(\alpha)\leq a}\mathbb{E}(\mathrm{Res}_{P_i}(\nabla),\,\alpha).$$

It also induces the filtration F of $\bigcup_h QC(E)(h\cdot D)$ (Lemma 3.12). The filtration is called the canonical filtration. It is well known that $\mu(QC(E),F)=0$ holds.

Lemma 3.13 (Sabbah) *Let (E,∇) be a flat connection on C. Then (E,∇) is simple if and only if $(QC(E),F,\nabla)$ is stable.*

Proof It is easy to see that the simplicity of (E,∇) implies the stability of $(QC(E),F,\nabla)$. Let us assume that (E,∇) is not simple. Then we have a sub-connection $(E',\nabla)\subset(E,\nabla)$. Then we obtain the filtered subbundle $\bigl(QC(E'),\nabla\bigr)\subset\bigl(QC(E),\nabla\bigr)$. Since we have $\mu(QC(E'),F)=0=\mu(QC(E),F)$, $\bigl(QC(E),F,\nabla\bigr)$ is not stable. Thus we are done. ∎

Corollary 3.2 *Let X be a projective variety, and D be a normal crossing divisor of X. Let (E,∇) be a flat connection on $X-D$. Let C be a smooth projective curve in X, which is transversal with D such that $\pi_1(C\setminus D)\longrightarrow\pi_1(X-D)$ is surjective (see Lemma 2.7).*

Assume that $\bigl(QC(E_{|C}),F,\nabla\bigr)$ is poly-stable, then (E,∇) is semisimple.

Proof Due to Lemma 3.13, we know that $(E,\nabla)_{|C}$ is semisimple. Since $\pi_1(C\setminus D)\longrightarrow\pi_1(X-D)$ is surjective, we obtain that (E,∇) is also semisimple. ∎

3.3.4 The end of the proof of Proposition 3.2

Let $(E,\overline{\partial}_E,\theta,h)$ be a tame harmonic bundle on a quasi projective curve C with the completion $\overline{C}$. We denote the corresponding flat bundle by $(\mathcal{E}^1,\mathbb{D}^1)$. Then we have the two kind of prolongment of $\mathcal{E}^1$ with the filtration. One prolongation is $QC(\mathcal{E}^1)$ with the canonical filtration. The other is the prolongment ${}^\diamond\mathcal{E}^1$ by an increasing order with the filtration with respect to the metric h.

Lemma 3.14 *If (E,∇,h) is pure imaginary, we have the canonical isomorphism $QC(\mathcal{E}^1)\simeq{}^\diamond\mathcal{E}^1$ preserving the filtrations.*

Proof It follows from Lemma 3.6 and the uniqueness of $QC(\mathcal{E}^1)$ ([8]). ∎

Let us show Proposition 3.2. Let $(E,\overline{\partial}_E,\theta,h)$ be a tame pure imaginary harmonic bundle on $X-D$. It is well known that we can take a smooth projective curve C in X such that $\pi_1(C\setminus D)\longrightarrow\pi_1(X-D)$ is surjective. Let us consider the restriction $(E,\overline{\partial}_E,\theta,h)_{|C\setminus D}$. Due to Lemma 3.14 and Proposition 3.3, we know that $QC(E)$ with the canonical filtration is polystable. Thus we obtain the semisimplicity of (E,∇) due to Corollary 3.2. Thus the proof of Proposition 3.2 is accomplished. ∎

3.4 The maximum principle

Let X be a compact Riemannian surface with the continuous boundary ∂X. Let U denote the interior part of X. Let us take $P_1, \ldots, P_l \in U$. We put $X^* := X - \{P_1, \ldots, P_l\}$ and $U^* := U - \{P_1, \ldots, P_l\}$. Let (E, ∇) be a flat bundle on X^*. A continuous hermitian metric h of (E, ∇) is called a tame pure imaginary harmonic bundle, if $(E, \nabla, h)_{|U^*}$ is tame pure imaginary harmonic bundle.

Let h_i $(i = 1, 2)$ be tame pure imaginary harmonic bundle of (E, ∇). The identity of E induces the flat morphisms $\Phi : (E, \nabla, h_1) \longrightarrow (E, \nabla, h_2)$. We obtain the norms $|\Phi|$ and $|\Phi^{-1}|$ obtained from h_1 and h_2.

Lemma 3.15 *The morphism Φ is bounded.*

Proof Let ${}^\diamond(E, h_i)$ denote the prolongment of E by an increasing order with respect to h_i. Due to Lemma 3.14, the morphism Φ is prolonged to the morphism ${}^\diamond(E, h_1) \longrightarrow {}^\diamond(E, h_2)$, which preserves the parabolic filtrations. Since the weight filtrations of ${}^\diamond(E, h_i)_{|P_i}$ are determined by the residue $\mathrm{Res}_{P_i}(\nabla)$, the morphism Φ also preserves the weight filtrations. Then it follows from the norm estimate of a tame harmonic bundle on a punctured disc. (See [39] or [33]). ∎

Lemma 3.16 *We have the following inequalities of the distributions on U:*

$$\Delta \log |\Phi|^2 \leq 0, \qquad \Delta \log |\Phi^{-1}|^2 \leq 0. \tag{32}$$

Proof If follows from the Simpson-Weitzenbeck formula and the boundedness of Φ and Φ^{-1} (see Lemma 4.1 and Corollary 4.2 in [39]). ∎

Lemma 3.17

- *$|\Phi|$ and $|\Phi^{-1}|$ take the maximum value at points in ∂X.*
- *We have the following inequality:*

$$|\Phi_{|P}|^2 + |\Phi^{-1}_{|P}|^2 - 2r \leq \max\Big\{ |\Phi_{|Q}|^2 + |\Phi^{-1}_{|Q}|^2 - 2r \,\Big|\, Q \in \partial X \Big\}.$$

∎

Lemma 3.18 *Let R be a real number such that $d_{\mathcal{PH}(r)}(h_{1\,|\,Q}, h_{2\,|\,Q}) \leq R$ for any point $Q \in \partial X$. Then the following inequality holds for any point $P \in X^*$:*

$$d_{\mathcal{PH}(r)}\big(h_{1\,|\,P}, h_{2\,|\,P}\big) \leq \left(\frac{e^R - e^{-R}}{2R}\right) \cdot \max\Big\{ d_{\mathcal{PH}(r)}(h_{1\,|\,Q}, h_{2\,|\,Q}) \,\Big|\, Q \in \partial X \Big\}.$$

Proof We always have the following:

$$d_{\mathcal{PH}(r)}(h_{1\,|\,P}, h_{2\,|\,P}) \leq \left(\frac{|\Phi_{|P}|^2 + |\Phi^{-1}_{|P}|^2 - 2r}{2} \right)^{1/2}.$$

For any point $Q \in \partial X$, we have the following:

$$\left(\frac{|\Phi_{|Q}|^2 + |\Phi^{-1}_{|Q}|^2 - 2r}{2} \right)^{1/2} \leq \frac{e^R - e^{-R}}{2R} \cdot d_{\mathcal{PH}(r)}\big(h_{1\,|\,Q}, h_{2\,|\,Q}\big).$$

Thus we are done. ∎

Corollary 3.3 *If we have $h_{1\,|\,\partial X} = h_{2\,|\,\partial X}$, we have $h_1 = h_2$.* ∎

3.5 The uniqueness of tame pure imaginary pluri-harmonic metric

3.5.1 The statement and the reduction to the one dimensional case

Let X be a smooth projective variety over $\boldsymbol{C}$ and D be a normal crossing divisor of X. Let (E, ∇) be a flat bundle over $X - D$. Let h_1 and h_2 be tame pure imaginary pluri-harmonic metric of (E, ∇).

Proposition 3.4 *Assume that* $\dim X \geq 1$. *There exists a positive constant a such that $h_0 = a \cdot h_1$.*

If X is compact, the claim is proved by Corlette [3]. We essentially follow his argument. Since we have to care the infinite energy, we need some modification of the argument.

We will prove the claim in the case $\dim X = 1$ later. Here we give an argument to reduce the higher dimensional case to the one dimensional case. We use an induction on $\dim X$. We assume the claim holds in the case $\dim X \leq n-1$, and we show that the claim holds in the case $\dim X = n$.

Let P be any point of X. We take a smooth hypersurface Y of X such that $P \in Y$ and $Y \cap D$ is normal crossing. Note $\dim(Y) = n-1 \geq 1$. Then there exists a positive constant a_Y such that $h_{0\,|\,Y} = a_Y \cdot h_{1\,|\,Y}$. In particular, there exists a positive constant $a(P)$ such that $h_{0\,|\,P} = a(P) \cdot h_{1\,|\,P}$.

For any point P and Q, we can take a smooth hypersurface Y such that $P, Q \in Y$ and $Y \cap D$ is normal crossing. Then we have $a(P) = a_Y = a(Q)$, i.e., $h_0 = h_1$.

Thus we have reduced the higher dimensional case to the one dimensional case. In the following subsubsections 3.5.2–3.5.5, we only consider a simple flat bundle (E, ∇) on a smooth quasi projective curve with the smooth projective completion $\overline{C}$.

3.5.2 Constantness of the eigenvalues

The hermitian metric h_1 induces the self adjoint morphism H of E with respect to the metric h_0. Let $\alpha_1, \ldots, \alpha_r$ be the eigenvalues of H, and then we have $|\Phi|^2 = \sum \alpha_i^2$.

Lemma 3.19 *We have the constantness of $\sum \alpha_i^2$ on C.*

Proof The identity map of E induces the flat morphism $\Phi : (E, \nabla, h_0) \longrightarrow (E, \nabla, h_1)$. Due to Lemma 3.16, we obtain the constantness of $|\Phi|$, i.e., the constantness of $\sum \alpha_i^2$. ∎

The tame pure imaginary harmonic metrics h_0 and h_1 induce those on $(\bigwedge^l E, \nabla)$ for any l. By applying Lemma 3.19, we obtain the constantness of any symmetric functions of α_i^2 $(i = 1, \ldots, r)$, which implies the constantness of α_i. Thus we obtain the following:

Lemma 3.20 *There exist the mutually different real numbers $\beta_1, \ldots, \beta_s$ and the decomposition $E = \bigoplus_i E_i$ satisfying the following:*

- *E_i are mutually orthogonal with respect to both of the metrics h_0 and h_1.*
- *On E_i, we have $h_1 = e^{2\beta_i} \cdot h_0$.* ∎

We put $\mathcal{L} = \bigoplus e^{-\beta_i} \cdot id_{E_i}$. Then we have $h_1(x,y) = h_0\big(\mathcal{L}^{-1}x, \mathcal{L}^{-1}y\big)$. We put as follows:

$$\mathcal{L}_t := \bigoplus e^{-t\beta_i} \cdot id_{E_i}, \qquad h_t(x,y) := h_0\big(\mathcal{L}_t^{-1}x, \mathcal{L}_t^{-1}y\big).$$

Let θ_t denote the $(1,0)$-form for (E, ∇, h_t) (see the subsubsection 2.5.3).

3.5.3 The description by connection forms

Let $\boldsymbol{e}$ be the orthonormal frame of E with respect to h_0 on some open subset of C with a coordinate z. The $(1,0)$-form $A^{1,0}dz$ and the $(0,1)$-form $A^{0,1}d\bar{z}$ are determined by $\nabla \boldsymbol{e} = \boldsymbol{e} \cdot \big(A^{1,0}dz + A^{0,1}d\bar{z}\big)$. We have the following:

$$d''\boldsymbol{e} = \boldsymbol{e} \cdot A^{0,1}d\bar{z}, \quad d'\boldsymbol{e} = \boldsymbol{e} \cdot A^{1,0}dz, \quad \delta''\boldsymbol{e} = \boldsymbol{e} \cdot \big(-\overline{{}^tA^{1,0}}\big)d\bar{z}, \quad \delta'\boldsymbol{e} = \boldsymbol{e} \cdot \big(-\overline{{}^tA^{0,1}}\big)dz.$$

Thus we have the following:

$$\theta_0 \boldsymbol{e} = \boldsymbol{e} \cdot \frac{A^{1,0} + \overline{{}^tA^{0,1}}}{2} dz, \quad \partial \boldsymbol{e} = \boldsymbol{e} \cdot \frac{A^{1,0} - \overline{{}^tA^{0,1}}}{2} dz. \tag{33}$$

Assume that $\boldsymbol{e}$ is compatible with the decomposition $E = \bigoplus E_i$. Then we have $\mathcal{L}_t \boldsymbol{e} = \boldsymbol{e} \cdot L_t$ for the constant diagonal matrices L_t. The frame $\boldsymbol{e} \cdot L_t$ is the orthonormal frame with respect to h_t. Via the frame $\boldsymbol{e} \cdot L_t$, we identify $End(E, E)$ with $M(r)$. We have the following:

$$\nabla\big(\boldsymbol{e} \cdot L_t\big) = \boldsymbol{e} \cdot L_t \cdot \big(L_t^{-1} A^{1,0} L_t \cdot dz + L_t^{-1} A^{0,1} L_t \cdot d\bar{z}\big).$$

Hence we have the following:

$$\theta_t\big(e \cdot L_t\big) = \boldsymbol{e} \cdot L_t \cdot \frac{\big(L_t^{-1} A^{1,0} L_t + L_t {}^t\overline{A^{0,1}} L_t^{-1}\big)}{2} dz.$$

The metric h_t induces the hermitian metric of $End(E)$. It induces the skew linear pairing

$$\langle \cdot, \cdot \rangle : \big(End(E) \otimes \Omega^{1,0}\big) \otimes \overline{\big(End(E) \otimes \Omega^{1,0}\big)} \longrightarrow \Omega^{1,1}.$$

We have the following formula:

$$\langle \theta_t, \theta_t \rangle = \frac{1}{4} \big|\big|L_t^{-1} A^{1,0} L_t + L_t {}^t\overline{A^{0,1}} L_t^{-1}\big|\big|^2 \cdot dz d\bar{z}.$$

Here $|| \cdot ||$ denote the norm of matrices. We have the decompositions:

$$A^{1,0} = \sum A^{1,0}_{i\,j}, \quad {}^t\overline{A^{0,1}} = \sum \big({}^t\overline{A^{0,1}}\big)_{i\,j}, \qquad \Big(A^{1,0}_{i\,j},\ \big({}^t\overline{A^{0,1}}\big)_{i\,j} \in Hom(E_i, E_j)\Big).$$

We also have the following:

$$L_t^{-1} A^{1,0} L_t = \sum_{i\,j} A^{1,0}_{i\,j} \cdot e^{(\beta_i - \beta_j)t}.$$

Thus we obtain the following:

$$\big|\big|L_t^{-1} A^{1,0} L_t + L_t {}^t\overline{A^{0,1}} L_t^{-1}\big|\big|^2 = \sum_{i\,j} \big|\big|A^{1,0}_{i\,j} \cdot e^{(\beta_i - \beta_j)t} + \big({}^t\overline{A^{0,1}}\big)_{i\,j} \cdot e^{-(\beta_i - \beta_j)t}\big|\big|^2.$$

3.5.4 The convexity

Note the following positivity, or convexity:

$$\begin{aligned} &\left(\frac{d}{dt}\right)^2 \big|\big|A^{1,0}_{i\,j} \cdot e^{(\beta_i - \beta_j)t} + {}^t\overline{A^{0,1}}_{i\,j} \cdot e^{-(\beta_i - \beta_j)t}\big|\big|^2 \\ &= 2(\beta_i - \beta_j)^2 \cdot \Big(\big|\big|A^{1,0}_{i\,j} \cdot e^{(\beta_i - \beta_j)t} + \big({}^t\overline{A^{0,1}}\big)_{i\,j} \cdot e^{-(\beta_i - \beta_j)t}\big|\big|^2 + \big|\big|A^{1,0}_{i\,j} \cdot e^{(\beta_i - \beta_j)t} - \big({}^t\overline{A^{0,1}}_{i\,j}\big) \cdot e^{-(\beta_i - \beta_j)t}\big|\big|^2\Big) \geq 0. \end{aligned} \tag{34}$$

Thus we obtain the following.

Lemma 3.21 *We have the following:*

$$\sqrt{-1} \left(\frac{d}{dt}\right)^2 \langle \theta_t, \theta_t \rangle \geq 0.$$

The equality holds if and only if $A^{1,0}_{i\,j} = A^{0,1}_{i\,j} = 0$ for any pair $i \neq j$. ∎

3.5.5 The end of the proof of Proposition 3.4

We have the line bundle $\mathcal{O}(D)$ on $\overline{C}$ with the canonical section $s : \mathcal{O} \longrightarrow \mathcal{O}(D)$. Let us take a C^∞-hermitian metric h_D of $\mathcal{O}(D)$. Then we obtain the C^∞-function $-\log|s|$ on C. We put $C(R) := \bigl\{P \in C \bigm| -\log|s(P)| \leq R\bigr\}$. Let us consider the following functions:

$$F_R(t) := \int_{C(R)} \sqrt{-1} \cdot \frac{d}{dt}\langle \theta_t, \theta_t \rangle, \qquad F_{R_1,R_2}(t) := \int_{C(R_1,R_2)} \sqrt{-1} \cdot \frac{d}{dt}\langle \theta_t, \theta_t \rangle.$$

Lemma 3.22 *We have $F_{R_1,R_2}(0) \leq F_{R_1,R_2}(1)$ for any R_1 and R_2. If $F_{R_1,R_2}(0) = F_{R_1,R_2}(1)$, then we obtain $(\frac{d}{dt})^2\langle \theta_t, \theta_t \rangle = 0$ on $C(R_1, R_2)$.*

Proof It follows from $\sqrt{-1} \cdot \bigl(\frac{d}{dt}\bigr)^2 \langle \theta_t, \theta_t \rangle \geq 0$. ∎

Corollary 3.4 *For any pair $R \geq R'$, we have $F_R(1) - F_R(0) \geq F_{R'}(1) - F_{R'}(0) \geq 0$.* ∎

Let us show that $F_R(0)$ converges to 0 when $R \to \infty$ by using the assumption that (E, ∇, h_0) is tame pure imaginary harmonic. We put $\xi := \frac{dL_t}{dt}$, which is self adjoint with respect to h_0. We have the following formula:

$$\frac{d}{dt}\Bigl(L_t^{-1}A^{1,0}L_t + L_t{}^t\overline{A^{0,1}}L_t^{-1}\Bigr)_{|t=0} = \left[\frac{A^{1,0} - {}^t\overline{A^{0,1}}}{2}, \xi\right] = \partial\xi.$$

Here we have used the formula (33). Thus we have the following:

$$\frac{d}{dt}\langle \theta_t, \theta_t \rangle_{|t=0} = \langle \partial\xi, \theta_0 \rangle + \langle \theta_0, \partial\xi \rangle = -\langle \xi, \overline{\partial}\theta_0 \rangle + \partial\langle \xi, \theta_0 \rangle + \langle \overline{\partial}\theta_0, \xi \rangle - \overline{\partial}\langle \theta_0, \xi \rangle = \partial\langle \xi, \theta_0 \rangle - \overline{\partial}\langle \theta_0, \xi \rangle.$$

Hence we obtain the following:

$$F_R(0) = \sqrt{-1}\int_{\partial C(R)} \Bigl(\langle \xi, \theta_0 \rangle - \langle \theta_0, \xi \rangle\Bigr).$$

Let P be a point of $\overline{C} - C$. We take a coordinate $z = re^{\sqrt{-1}\eta}$ around P. Then we have the description:

$$\theta_0 = g \cdot \frac{dz}{z} = g\left(\frac{dr}{r} + \sqrt{-1}d\eta\right).$$

Here g is an endomorphism of E. We recall that the eigenvalues of $g_{|P}$ is pure imaginary. Due to Simpson's Main estimate, we have the decomposition $g = g_0 + g_1$ (see the subsection 7.1 in [33]):

- There is a decomposition $E = \bigoplus_{\alpha \in \sqrt{-1}\boldsymbol{R}} E'_\alpha$ which is orthogonal with respect to h_0. The endomorphism g_0 is given by $\bigoplus \alpha\, \mathrm{id}_{E'_\alpha}$.
- We have the estimate $|g_1| \leq C \cdot (-\log r)^{-1}$ for some positive constant C.

Since $\sqrt{-1} \cdot g_0$ is self adjoint, we have the cancellation: $\bigl\langle \xi, \sqrt{-1}g_0 \cdot d\eta \bigr\rangle - \bigl\langle \sqrt{-1}g_0 \cdot d\eta, \xi \bigr\rangle = 0$. We also have the estimate $\bigl|\langle \xi, \sqrt{-1}g_1 \rangle\bigr| \leq C \cdot (\log R)^{-1}$ for some positive constant C. Then we obtain that $\lim_{R\to\infty} F_R(0) = 0$.

Lemma 3.23 *We have the convergence* $\lim_{R\to\infty}\bigl(F_R(1) - F_R(0)\bigr) = 0$. ∎

Proof We have only used the property that (E, ∇, h_0) is tame pure imaginary harmonic to show $\lim_{R\to\infty} F_R(0) = 0$. Hence we also obtain $\lim_{R\to\infty} F_R(1) = 0$. Then the lemma immediately follows. ∎

We obtain $F_R(1) - F_R(0) = 0$ for any R, from Corollary 3.4 and Lemma 3.23. It implies $(\frac{d}{dt})^2\langle \theta_t, \theta_t \rangle = 0$ for any $t \in [0, 1]$, and thus $A^{0,1}_{i\,j} = A^{1,0}_{i\,j} = 0$ for $i \neq j$, due to Lemma 3.22. It implies that the decomposition $E = \bigoplus_i E_i$ is flat with respect to the connection ∇.

Since E is simple, we have $E = E_i$ for some i, and thus we obtain $h_1 = e^{2\beta_i} \cdot h_0$. Therefore the proof of Proposition 3.4 is accomplished. ∎

4 Depencence on boundary value in the case of a punctured disc

4.1 The Dirichlet problem and sequence of the boundary values

4.1.1 The Dirichlet problem

Let φ be an element of $GL(r)$, and (E,∇) be a flat bundle on $\overline{\Delta}^*$ whose monodromy is conjugate with φ. Let $h_{\partial\overline{\Delta}}$ be a C^{∞}-hermitian metric on $(E,\nabla)_{|\partial\overline{\Delta}}$. We denote the corresponding C^{∞}-map $\partial\overline{\Delta} \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$ by ψ.

Proposition 4.1 *There exists the tame pure imaginary harmonic metric h on (E,∇) such that $h_{|\partial\overline{\Delta}} = h_{\partial\overline{\Delta}}$. It is unique up to constant multiplication.*

Proof The uniqueness immediately follows from Corollary 3.3. Hence we have only to show the existence. The main idea is clearly explained in [29] and [23].

We put $T(R_1,R_2) := \bigl\{z \in \boldsymbol{C} \,|\, R_1 \leq -\log|z| \leq R_2\bigr\}$, and $T(R) := T(0,R)$. We use the Poincaré metric $g := |z|^{-2}\cdot\bigl(-\log|z|^2 + A\bigr)^{-2}\cdot dz\cdot d\bar{z}$ of $\overline{\Delta}^*$ for some positive constant A. We also use the real coordinate $z = \exp\bigl(\sqrt{-1}x - y\bigr)$.

Lemma 4.1 *We have a C^{∞}-hermitian metric h_0 on $\overline{\Delta}^*$ satisfying the following:*

$$h_{0\,|\,\partial\overline{\Delta}} = h_{\partial\overline{\Delta}}, \qquad \int_{\overline{\Delta}^*}\left|e(\Psi_{h_0}) - \frac{\rho(\varphi)^2}{4\pi^2}\bigl(2y+A\bigr)^2\right|\cdot \mathrm{dvol}_g < \infty.$$

Proof Recall that we have a tame pure imaginary harmonic metric of (E,∇) on $\overline{\Delta}^*$, by using the model bundle for example (see [33]). Hence we can take a C^{∞}-hermitian metric h_0 of (E,∇) satisfying the following:

- $h_{0\,|\,\partial\overline{\Delta}} = h_{\partial\overline{\Delta}}$.
- The restriction of h_0 to $\overline{\Delta}^* - T(1)$ is a tame pure imaginary harmonic metric.

Due to Lemma 3.8, we are done. ∎

Remark 4.1 *We can avoid to use the model bundle as in* [23]. *See also Lemma* 4.13*, for example.* ∎

The function F is given as follows, which is integrable with respect to dvol_g:

$$F := \left|e\bigl(\Psi_{h_0}\bigr) - \frac{\rho(\varphi)^2}{4\pi^2}\cdot(2y+A)^2\right|. \tag{35}$$

Lemma 4.2 (Hamilton-Schoen-Corlette [6]) *There exists the twisted harmonic map $\Phi_n : T(n) \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$ satisfying the following:*

$$\Phi_{n\,|\,\partial\overline{\Delta}} = \psi, \qquad \Phi_{n\,|\,|z|=e^{-n}} = \Psi_{h_0\,|\,|z|=e^{-n}}.$$

$$\int_{T(n)} e(\Phi_n)\,\mathrm{dvol}_g \leq \int_{T(n)} e(\Psi_{h_0})\,\mathrm{dvol}_g\,.$$

Proof See the proof of Theorem 2.1 of [6]. ∎

Lemma 4.3 *Let R be a positive number. For any $n > R$, we have the following inequality:*

$$\int_{T(R)} e(\Phi_n)\cdot\mathrm{dvol}_g \leq R\cdot\frac{\rho(\varphi)^2}{2\pi} + \int_{\overline{\Delta}^*} F\cdot\mathrm{dvol}_g\,.$$

Proof We have the following inequalities:

$$\int_{T(n)} e\bigl(\Psi_{h_0}\bigr)\cdot \mathrm{dvol}_g \geq \int_{T(n)} e\bigl(\Phi_n\bigr)\cdot \mathrm{dvol}_g = \int_{T(R)} e\bigl(\Phi_n\bigr)\cdot \mathrm{dvol}_g + \int_{T(R,n)} e\bigl(\Phi_n\bigr)\cdot \mathrm{dvol}_g$$
$$\geq \int_{T(R)} e\bigl(\Phi_n\bigr)\cdot \mathrm{dvol}_g + \frac{\rho(\varphi)^2}{2\pi}\cdot (n-R). \quad (36)$$

Here we have used Lemma 3.9. On the other hand, we have the following inequality by our choice of the integrable function F (35):

$$\int_{T(n)} e(\Psi_{h_0})\cdot \mathrm{dvol}_g \leq \frac{\rho(\varphi)^2}{2\pi}\cdot n + \int_{T(n)} F\cdot \mathrm{dvol}_g\,. \quad (37)$$

From (36) and (37), we obtain the following:

$$\int_{T(R)} e(\Phi_n)\cdot \mathrm{dvol}_g \leq R\cdot \frac{\rho(\varphi)^2}{2\pi} + \int_{T(n)} F\cdot \mathrm{dvol}_g\,.$$

Thus we are done. ∎

We have the projection $p:\overline{\Delta}^{*}\longrightarrow \partial\overline{\Delta}$ given by $p(e^{\sqrt{-1}x-y})=e^{\sqrt{-1}x}$. We have the following, for any point $P=e^{\sqrt{-1}x_0-y_0}\in T(R)$:

$$d_{\mathcal{PH}(r)}\bigl(\Phi_n(P),\psi(p(P))\bigr)=d_{\mathcal{PH}(r)}\bigl(\Phi_n(P),\Phi_n(p(P))\bigr)\leq \int_0^{y_0}|\partial_y\Phi_n|\cdot dy\leq \int_0^R |\partial_y\Phi_n|\cdot dy$$
$$\leq C_R\cdot\left(\int_0^R |\partial_y\Phi_n|^2\cdot dy\right)^{1/2}\leq C_R\cdot\left(\int_0^R e(\Phi_n)\cdot\frac{dy}{(2y+A)^2}\right)^{1/2}. \quad (38)$$

Hence for any subregion $B\subset T(R)$, there exists a positive constant $C_{R,B}$, which is independent of a choice of n, such that the following inequalities hold:

$$\int_B d_{\mathcal{PH}(r)}\bigl(\Phi_n(P),\psi(p(P))\bigr)^2\cdot d\eta\cdot dr\leq C_{R,B}\cdot\int_{T(R)} e(\Phi_n)\cdot \mathrm{dvol}_g\,. \quad (39)$$

Lemma 4.4 *Let $\boldsymbol{n}_0$ be any infinite subset of $\boldsymbol{N}$. Then there exist an infinite subset $\boldsymbol{n}_1\subset\boldsymbol{n}_0$ such that the sequence $\bigl\{\Phi_n\,\big|\,n\in\boldsymbol{n}_1\bigr\}$ is C^{∞}-convergent on any compact subset $K\subset\Delta^*$.*

Proof From (39), we can take a point $P\in\Delta^*$ and a subsequence $\boldsymbol{n}_2$ such that $\bigl\{\Phi_n(P)\,|\,n\in\boldsymbol{n}_2\bigr\}$ is convergent. We have the boundedness of $e(\Phi_n)$ on any compact subset $K\subset\Delta^*$, which can be derived from the boundedness of the energy $E(\Phi_n)$ (Lemma 4.3) and Lemma 2.25. Then it is standard to show the result by using the boot strapping argument. (See [29] and [37]). ∎

Let us fix the subsequence $\boldsymbol{n}_1$ for which $\{\Phi_n\,|\,n\in\boldsymbol{n}_1\}$ is convergent on any compact subset $K\subset\Delta^*$. Let Φ_∞ denote the limit, and let h denote the corresponding harmonic metric of (E,∇). We also use the notation Ψ_h to denote Φ_∞.

Lemma 4.5 *The sequence $\bigl\{\Phi_n\,\big|\,n\in\boldsymbol{n}_1\bigr\}$ is C^0-convergent on $T(R)$ for any R. In particular, $\Phi_{\infty\,|\,\partial\Delta}=\psi$.*

Proof Due to our choice of $\boldsymbol{n}_1$, the sequence $\bigl\{\Phi_{n\,|\,|z|=e^{-k}}\,|\,n\in\boldsymbol{n}_1\bigr\}$ is C^{∞}-convergent. By our choice of Φ_n, we have $\Phi_{n\,|\,\partial\Delta}=\psi$. Then we obtain the C^0-convergence due to Lemma 3.18. ∎

Due to Lemma 4.3, we obtain the following inequality:

$$\int_{T(R)} e\bigl(\Psi_h\bigr)\cdot \mathrm{dvol}_g\leq R\cdot\frac{\rho(\varphi)^2}{2\pi}+C.$$

Here C denotes a positive constant, which is independent of R. Then it is easy to obtain an integrable function $\tilde{F}$ on $\overline{\Delta}^*$, satisfying the following inequalities for any sufficiently large real number R:

$$\int_{T(R)} e\big(\Psi_h\big) \cdot \mathrm{dvol}_g \leq \int_{T(R)} \left(\frac{\rho(\varphi)^2}{4\pi^2} \cdot \big(2y + A\big)^2 + \tilde{F} \right) \cdot \mathrm{dvol}_g \, .$$

Therefore the harmonic bundle (E, ∇, h) is tame pure imaginary due to Proposition 3.1. Thus the proof of Proposition 4.1 is accomplished. ∎

We give a remark on the convergency of the sequence $\big\{\Phi_n\big\}$.

Lemma 4.6 *The sequence $\big\{\Phi_n\big\}$ is C^∞-convergent on any compact subset $K \subset \Delta^*$, and it is C^0-convergent on $T(R)$ for any R.*

Proof Let $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ be as in Lemma 4.4. We have the limit Φ_∞. It is tame pure imaginary harmonic. It also satisfies $\Phi_{\infty\,|\,\partial\Delta} = \psi$. Then it follows that Φ_∞ is independent of a choice of $\boldsymbol{n}_1$, due to Corollary 3.3. It implies the desired convergence properties of the sequence $\big\{\Phi_n \,\big|\, n \in \boldsymbol{N}\big\}$. ∎

4.1.2 Dependence of a convergent sequence of boundary values

Let $\{\varphi_i\}$ be a sequence in $GL(r)$ converging to A. Let $\Phi_i : \overline{\Delta}^* \longrightarrow \mathcal{PH}(r)\big/\langle\varphi_i\rangle$ and $\Phi : \overline{\Delta}^* \longrightarrow \mathcal{PH}(r)\big/\langle\varphi\rangle$ be maps, which are corresponding to tame pure imaginary harmonic bundles on $\overline{\Delta}^*$.

Proposition 4.2 *Assume the following:*

- *The sequence of the boundary values $\big\{\Phi_{i\,|\,\partial\overline{\Delta}}\big\}$ converges to $\Phi_{|\partial\overline{\Delta}}$ in C^∞-sense.*
- *There exists a positive constant C, which is independent of i and k, satisfying the following:*

$$\int_{T(k)} e(\Phi_{i\,|\,T_k}) \cdot \mathrm{dvol}_g \leq k \cdot \frac{\rho(\varphi_i)^2}{2\pi} + C.$$

Then the sequence $\{\Phi_i\}$ is C^∞-convergent on any compact subset $K \subset \Delta^*$, and C^0-convergent on any T_k.

Proof The argument is essentially same as the proof of Proposition 4.1 and Lemma 4.6. Hence we only indicate an outline.

Let $\boldsymbol{n}_0$ be an infinite subset of $\boldsymbol{N}$. Then we can take an infinite subset $\boldsymbol{n}_1 \subset \boldsymbol{n}_0$ such that the sequence $\{\Phi_i \,|\, n \in \boldsymbol{n}_1\}$ is C^∞-convergent on any compact subset $K \subset \Delta^*$, by an argument similar to the proof of Lemma 4.4. We denote the limit by Φ_∞. We can show that the sequence $\big\{\Phi_i \,\big|\, n \in \boldsymbol{n}_1\big\}$ is C^0-convergent on any T_k by an argument similar to the proof of Lemma 4.5. In particular, we obtain $\Phi_{\infty\,|\,\partial\overline{\Delta}} = \Phi_{|\partial\overline{\Delta}}$. By using the estimate of the energy and Proposition 3.1, we can show that Φ_∞ corresponds to the tame pure imaginary harmonic bundle. Thus we obtain $\Phi_\infty = \Phi$ from the maximum principle (Lemma 3.18). Since Φ_∞ is independent of a choice of $\boldsymbol{n}_1$, we obtain the desired convergency of $\big\{\Phi_i\big\}$. ∎

4.2 Family version

4.2.1 Estimate for a flat family

Let X be a C^∞-manifold. Let (E, ∇) be a flat bundle on $\overline{\Delta}^* \times X$. Let h_1 be a C^∞ hermitian metric of $(E, \nabla)_{|\partial\overline{\Delta}\times X}$. We have the hermitian metric h on (E, ∇) such that $(E, \nabla, h)_{|\overline{\Delta}^*\times P}$ are tame pure imaginary harmonic bundles on $\overline{\Delta}^*$ for any $P \in X$.

Let us pick any point P_0 of X, and take an appropriate neighbourhood U of P_0. For any point $P \in U$, we may assume to have the natural identification $(E, \nabla)_{|\overline{\Delta}^*\times P} \simeq (E, \nabla)_{|\overline{\Delta}^*\times P_0}$. Hence we obtain the family of the maps $\phi_P : \overline{\Delta}^* \times \{P_0\} \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$. Here φ denotes the monodromy.

Lemma 4.7 *We regard h as the family of tame pure imaginary harmonic metrics $\big\{h_{\overline{\Delta}^*\times P} \,\big|\, P \in X\big\}$. Then the assumption in Proposition 4.2 is satisfied.*

Proof It is easy to see that we can take a C^∞-hermitian metric h_2 of $(E,\nabla)_{|\overline{\Delta}^*\times U}$ such that the following holds:

- $h_{2\,|\,\partial\overline{\Delta}\times U} = h_{1\,|\,\partial\overline{\Delta}\times U}$,
- The restriction of h_2 to $\overline{\Delta}^* - S_1$ is the fixed tame pure imaginary harmonic metric, i.e., $h_{2\,|\,(\overline{\Delta}^*-S_1)\times\{P\}} = h_{2\,|\,(\overline{\Delta}^*-S_1)\times\{P_0\}}$ under the isomorphism $(E,\nabla)_{|\overline{\Delta}^*\times\{P\}} \simeq (E,\nabla)_{|\overline{\Delta}^*\times\{P_0\}}$.

Let $\Psi_{h_2\,P} : \overline{\Delta}^* \times \{P_0\} \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$ denote the map corresponding to $h_{2\,|\,\overline{\Delta}^*\times P}$. There exists a positive constant C, which is independent of a choice of P, such that the following holds:

$$\int_{\overline{\Delta}^*} \left| e(\Psi_{h_2\,P}) - \frac{\rho(\varphi)^2}{4\pi^2}\cdot\bigl(-\log|z|^2 + A\bigr)^2 \right| \cdot \mathrm{dvol} < C.$$

Then the claim of Lemma 4.7 is clear from the proof of Proposition 4.1. ∎

Lemma 4.8 *The corresponding map Ψ_h is continuous. We also have the continuity of $\partial_z^l \overline{\partial}_z^m \Psi_h$ for any l and m.*

Proof It follows from Proposition 4.2 and Lemma 4.7. ∎

For any point $P \in X$, we put as follows:

$$(E^P, \nabla^P, h^P) := (E, \nabla, h)_{|\overline{\Delta}^*\times P}.$$

We obtain $\overline{\partial}_{E^P}$ and θ^P.

Lemma 4.9 $\bigl\{\overline{\partial}_{E^P} \bigm| P \in X\bigr\}$ *and* $\bigl\{\theta^P \bigm| P \in X\bigr\}$ *are continuous with respect to P.*

Proof It follows from Lemma 4.8. ∎

We have the description $\theta^P = f_0^P \cdot dz/z$. We obtain the family of the characteristic polynomial:

$$\det(t - f_0^P) = \sum t^i \cdot A_i(z,P).$$

Here $A_i(z,P)$ is continuous on $\overline{\Delta}^* \times X$. The restrictions of A_i to $\overline{\Delta}^* \times P$ are holomorphic, and it is naturally extended to the holomorphic functions on $\overline{\Delta} \times P$. Then it is easy to see that A_i is naturally extended to the continuous function on $\overline{\Delta} \times X$, by using the Cauchy's integral formula.

For any point $P \in X$, we can take positive constants $R(P), C(P), \epsilon(P)$ as in Lemma 3.3.

Lemma 4.10 *We can take the constants $R(P), C(P), \epsilon(P)$ independently of P.*

Proof It follows from the continuity of A_i on $\overline{\Delta} \times X$. ∎

Lemma 4.11 *Positive constants R', C', ϵ' in Lemma 3.4, a positive constant C'' in Lemma 3.5, a positive constant C_1 in Lemma 3.8 and a positive constant C_0 in Corollary 3.1 for $(E^P, \overline{\partial}_{E^P}, h^P, \theta^P)$ can be taken independently of P.* ∎

Let $(x_1, \ldots, x_l)$ be a coordinate of X. Recall that $\partial\Psi_h/\partial x_i$ are defined almost everywhere.

Lemma 4.12 *For any point $T \in X$, we have the following:*

$$\left|\frac{\partial\Psi_h}{\partial x_i}\right|(P,T) \leq \max\left\{\left|\frac{\partial\Psi_h}{\partial x_i}\right|(Q,T) \Bigm| Q \in \partial\overline{\Delta}\right\}$$

Proof It follows from Lemma 3.18. ∎

4.2.2 Another family

Recall we have the continuous family of C^∞-twisted maps $\{\tilde{\Phi}_{(\xi_1,\kappa)} \,|\, (\xi_1,\kappa) \in \boldsymbol{R} \times [0,1]\}$ (see the subsubsection 2.4.4).

Lemma 4.13 *For any $(\xi_1,\kappa) \in \boldsymbol{R} \times [0,1]$, we have twisted harmonic maps $\Psi_{h\,(\xi_1,\kappa)} : \overline{\Delta}^* \longrightarrow \mathcal{PH}(r)/\langle \tilde{f}_2(\kappa)\rangle$ which satisfy the following:*

- $\Psi_{h\,(\xi_1,\kappa)\,|\,\partial\overline{\Delta}} = \tilde{\Phi}_{(\xi_1,\kappa)\,|\,\partial\overline{\Delta}}$.
- *There exists a positive constant C, which is independent of a choice of (ξ_1,κ), such that the following holds:*

$$\int_{\overline{\Delta}^*} \left| e\big(\tilde{\Psi}_{h\,(\xi_1,\kappa)}\big) - \frac{\rho(f_2)^2}{4\pi^2} \cdot (2\eta_2 + b)^2 \right| \mathrm{dvol} < C. \tag{40}$$

Proof We have the finiteness as in Lemma 2.24. We can take $\Phi_{(\xi_1,\kappa)}$ as the hermitian metric in Lemma 4.1, and we construct the twisted harmonic maps $\Psi_{h\,(\xi_1,\kappa)}$ as in the subsection 4.1. Then we obtain the estimate of the integral (40), independently of (ξ_1,κ). ∎

Corollary 4.1 *For any non-negative integers l and m, the family $\{\partial^m_{\xi_2}\partial^l_{\eta_2}\Psi_{h,(\xi_1,\kappa)} \,|\, (\xi_1,\kappa) \in \boldsymbol{R} \times [0,1]\}$ is continuous with respect to (ξ_1,κ).* ∎

From the family $\{\Psi_{h\,(\xi_1,\kappa)} \,|\, (\xi_1,\kappa) \in \boldsymbol{R} \times [0,1]\}$, we obtain the continuous family of the flat bundles $(E,\nabla)_{(\xi_1,\kappa)}$ with the tame pure imaginary harmonic metrics $h_{(\xi_1,\kappa)}$ on $\overline{\Delta}^*$. We obtain the Higgs fields $\theta_{(\xi_1,\kappa)}$. Similarly to Lemma 4.10 and Lemma 4.11, we obtain the following.

Lemma 4.14 *We can take the constants $R(\xi_1,\kappa), C(\xi_1,\kappa), \epsilon(\xi_1,\kappa)$ in Lemma 3.3 independently of (ξ_1,κ).* ∎

Lemma 4.15 *Positive constants R', C', ϵ' in Lemma 3.4, a positive constant C'' in Lemma 3.5, a positive constant C_1 in Lemma 3.8 and a positive constant C_0 in Corollary 3.1 for $(E, \overline{\partial}_E, h, \theta)_{(\xi_1,\kappa)}$ can be taken independently of (ξ_1,κ).* ∎

5 Control of the energy of twisted map on a Kahler surface

5.1 Around smooth points of divisors

5.1.1 Metric

Let X be a closed region of $\boldsymbol{C}^l$ whose boundary is continuous. We consider the set $X \times \Delta(R) \subset \boldsymbol{C}^l \times \boldsymbol{C}$. We use the real coordinate $(x_1,\ldots,x_{2l})$ of $\boldsymbol{C}^l$. We denote $\partial/\partial x_i$ also by ∂_i.

We use the real coordinate $z = x + \sqrt{-1}y$ of $\boldsymbol{C}$. We also use the polar coordinate $z = r \cdot e^{\sqrt{-1}\eta}$ of $\boldsymbol{C}$. We denote $\partial/\partial r$ and $\partial/\partial\eta$ also by ∂_r and ∂_η respectively. We denote the origin of $\boldsymbol{C}$ by O.

We have the natural complex structure J_X of X, which is induced by the complex structure of $\boldsymbol{C}^l$. For any point $P \in X$, we denote $\{P\} \times \Delta(R)$ by $P \times \Delta(R)$ for simplicity. We have the natural complex structure $J_{P\times\Delta(R)}$ on $P \times \Delta(R)$, which is induced by the complex structure of $\boldsymbol{C}$.

Let J be a complex structure of $X \times \Delta(R)$ satisfying the following conditions:

Condition 5.1

- *The natural inclusion $X \times \{O\} \longrightarrow X \times \Delta(R)$ is a holomorphic embedding.*
- *Let P be any point of X. We have the natural inclusion $T_{(P,O)}(P \times \Delta(R)) \longrightarrow T_{(P,O)}(X \times \Delta(R))$ of the tangent spaces. Then $T_{(P,O)}(P \times \Delta(R))$ is a subspace of $\boldsymbol{C}$-vector space $T_{(P,O)}(X \times \Delta(R))$, and the restriction of J to $T_{(P,O)}(P \times \Delta(R))$ is same as $J_{P\times\Delta(R)}$.*

Remark 5.1 *The complex structure J is not necessarily same as the natural complex structure of $X \times \Delta(R)$.* ∎

Let P be any point of X. Let ζ be a holomorphic function of $(X \times \Delta(R), J)$ defined on a neighbourhood U of (P, O) satisfying the following:

$$d\zeta_{|(P,O)} \neq 0, \quad \zeta^{-1}(0) = U \cap (X \times \{O\}).$$

Let z be the holomorphic coordinate of $\boldsymbol{C}$. It induces a C^∞-function on $X \times \Delta(R)$, which is not necessarily holomorphic with respect to the complex structure J.

Lemma 5.1 *There exists the complex number $a(P)$ such that $dz_{|(P,O)} = a(P) \cdot d\zeta_{|(P,O)}$.*

Proof Both of $d\zeta_{|(P,O)}$ and $dz_{|(P,O)}$ vanish on $T(X \times \{O\})$. Hence we have only to compare them on the one dimensional subspace $T_{(P,O)}(P \times \Delta(R))$. Since both of $d\zeta_{|(P,O)}$ and $dz_{|(P,O)}$ are $\boldsymbol{C}$-linear, we obtain the complex number $a(P)$ as claimed. ∎

For simplicity, we consider the case ζ is defined over $X \times \overline{\Delta}$. We can take a C^∞-function $a : X \times \overline{\Delta} \longrightarrow \boldsymbol{C}^*$ such that $(a \cdot d\zeta)_{|X \times \{O\}} = dz_{|X \times \{O\}}$. We have the following estimates:

$$|a\zeta|^2 = |z|^2 + O(|z|^3), \qquad \left| d\zeta\left(\frac{\partial}{\partial x_i}\right)\right| = O\big(|z|\big). \tag{41}$$

Let g_1 be a Kahler metric of $(X \times \overline{\Delta}, J)$, and ω_1 be the associated Kahler form. We have the following estimates with respect to the metric g_1:

$$\big|dz - d(a\zeta)\big|_{g_1} = O(|z|), \quad \frac{dz}{z} - \frac{d\zeta}{\zeta} = O(1). \tag{42}$$

Let b be any C^∞-function on $X \times \overline{\Delta}$ such that $-\log|\zeta|^2 + b > 0$ on $X \times \overline{\Delta}^*$ and $\big(b + \log|a|^2\big)_{|X \times \{O\}} > 0$ on $X \times \{O\}$. We put $A := \big(b + \log|a|^2\big)_{X \times \{O\}}$. We have the estimate:

$$\Big|(-\log|z|^2 + A) - (-\log|\zeta|^2 + b)\Big| = O(|z|). \tag{43}$$

We have the following formula:

$$\begin{aligned}\partial\overline{\partial}\log\Big(-\log|\zeta|^2 + b\Big) &= \partial\left(\frac{\overline{\partial}\big(-\log|\zeta|^2 + b\big)}{-\log|\zeta|^2 + b}\right) \\ &= -\frac{1}{\big(-\log|\zeta|^2 + b\big)^2}\left(\frac{d\zeta}{\zeta} - \partial b\right) \wedge \left(\frac{d\bar{\zeta}}{\bar{\zeta}} - \overline{\partial} b\right) + \frac{-\partial\overline{\partial} b}{(-\log|\zeta|^2 + b)}.\end{aligned} \tag{44}$$

Assume that the following gives the Kahler form:

$$\omega = \omega_1 - \sqrt{-1}\partial\overline{\partial}\log\Big(-\log|\zeta|^2 + b\Big).$$

Let g denote the associated Kahler metric. Let g_2 (resp. g_3) denote the hermitian form corresponding to the first (resp. second) term in the right hand side of (44). Thus we have the decomposition $g = g_1 + g_2 + g_3$.

Recall we use the polar coordinate $z = r \cdot e^{\sqrt{-1}\eta}$.

Lemma 5.2 *We have the following estimates:*

$$\big|\partial_r\big|_g^2 = (1 + F_1) \cdot \frac{1}{r^2 \cdot (-\log r^2 + A)^2}, \qquad |F_1| = O\big(r \cdot (-\log r^2 + A)\big).$$

$$\big|\partial_\eta\big|_g^2 = (1 + F_2) \cdot \frac{1}{(-\log r^2 + A)^2}, \qquad |F_2| = O\big(r \cdot (-\log r^2 + A)\big).$$

Proof It follows from (42), (43) and (44). Note $|\partial_\eta|_{g_1} = O(r)$. ∎

Lemma 5.3 *We have the following estimate:*

$$\big|g(\partial_r,\partial_\eta)\big| = O\big((-\log r^2 + A)^{-2}\big), \qquad \left|\frac{g(\partial_r,\partial_\eta)}{|\partial_\eta|_g^2}\right| = O(1).$$

Proof We have the following estimates:

$$\big|g_1\big(\partial_r,\partial_\eta\big)\big| = O(r), \qquad \big|g_3\big(\partial_r,\partial_\eta\big)\big| = O\big(r\cdot(-\log r^2 + A)^{-1}\big).$$

Since ∂_r and ∂_η are orthogonal with respect to the hermitian form $dz\cdot d\bar{z}$, we obtain the following estimate from (42).

$$\big|g_2\big(\partial_r,\partial_\eta\big)\big| = O\Big(\big(-\log r^2 + A\big)^{-2}\Big).$$

Then we obtain the result from (44). ∎

We put as follows:

$$e_1 := \partial_\eta, \qquad e_2 := \partial_r - \frac{g(\partial_r,\partial_\eta)}{|\partial_\eta|_g^2}\cdot\partial_\eta.$$

Lemma 5.4 *We have the following estimate:*

$$|e_2|_g^2 = (1+F_3)\cdot\frac{1}{r^2(-\log r^2 + A)^2}, \qquad |F_3| = O\big(r\cdot(-\log r^2 + A)\big).$$

Proof We have $|e_2|_g^2 = |\partial_r|_g^2 - |g(\partial_r,\partial_\eta)|^2\cdot|\partial_\eta|_g^{-2}$. Then it is easy to check the claim. ∎

We use the notation ∂_i to denote $\partial/\partial x_i$.

Lemma 5.5 *We have the following estimate:*

$$g\big(\partial_i,e_1\big) = O\big((-\log r^2 + A)^{-2}\big), \qquad g\big(\partial_i,e_2\big) = O\big(r^{-1}\cdot(-\log r^2 + A)^{-2}\big).$$

Proof We have estimates $(g_1+g_3)\big(\partial_i,e_1\big) = O(r)$. We also have $g_2\big(\partial_i,e_1\big) = O\big((-\log r^2 + A)^{-2}\big)$, which follows from (42). Thus we obtain the first estimate. Similarly, we can obtain the second estimate. ∎

We put as follows:

$$f_i := \partial_i - \frac{g(\partial_i,e_1)}{|e_1|_g^2}\cdot e_1 - \frac{g(\partial_i,e_2)}{|e_2|_g^2}\cdot e_2.$$

Lemma 5.6 *We have the following estimate:*

$$\big|g\big(f_i,f_j\big) - g_1(\partial_i,\partial_j)\big| = O\big((-\log r^2 + A)^{-1}\big).$$

Proof We have the following:

$$g(f_i,f_j) = g(\partial_i,\partial_j) - \sum_{a=1,2}\frac{g(\partial_i,e_a)\cdot g(\partial_j,e_a)}{|e_a|_g^2}.$$

The second term is $O\big((-\log r^2 + A)^{-2}\big)$, which follows from the previous lemmas. It is easy to obtain the following estimate, from (44):

$$g(\partial_i,\partial_j) = g_1(\partial_i,\partial_j) + O\big((-\log r^2 + A)^{-1}\big).$$

Thus we are done. ∎

5.1.2 The volume form

The restriction of g_1 to $X \times \{0\}$ induces the volume form of X, which we denote by dvol_X.

Lemma 5.7 *Let $\pi : X \times \overline{\Delta} \longrightarrow X$ be the projection. We have the following estimate:*

$$\mathrm{dvol}_g = (1 + F_4) \cdot \pi^* \mathrm{dvol}_X \cdot \frac{dr \cdot d\eta}{r \cdot (-\log r^2 + A)^2}, \qquad |F_4| = O\big((-\log r^2 + A)^{-1}\big). \tag{45}$$

Proof We have the naturally defined subbundle $T\overline{\Delta} \subset T(X \times \overline{\Delta})$. Let H be the orthogonal complement of $T\overline{\Delta}$ in $T(X \times \overline{\Delta})$. The bundle H is generated by $f_1, \ldots, f_{2l}$.

The metric g induces the volume form of $T\overline{\Delta}$, which is as follows, due to Lemma 5.2 and Lemma 5.4:

$$(1 + G_1) \cdot \frac{dr \cdot d\eta}{r \cdot (-\log r^2 + A)^2}, \qquad |G_1| = O\big(r \cdot (-\log r^2 + A)\big).$$

On the other hand, we have the volume form of H induced by g, which is as follows, due to Lemma 5.6:

$$(1 + G_2) \cdot \pi^* \mathrm{dvol}_X, \qquad |G_2| = O\big((-\log r^2 + A)^{-1}\big).$$

Then (45) follows. ∎

5.1.3 An inequality for energy

Let Y be a Riemannian manifold on which $\pi_1(X \times \overline{\Delta}^*)$ acts. Let $F : X \times \overline{\Delta}^* \longrightarrow Y \big/ \pi_1(X \times \overline{\Delta}^*)$ be a twisted map. Let $e_g(F)$ denote the energy function of F with respect to the metric g of X.

Lemma 5.8 *There exists a positive number C such that the following holds for any twisted map F:*

$$\left| e_g(F) - (-\log r^2 + A)^2 \cdot \left| \frac{\partial F}{\partial \eta} \right|^2 \right| \leq C \cdot \left(\sum_{i=1}^{l} \left| \frac{\partial F}{\partial x_i} \right|^2 + r^2 \cdot (-\log r^2 + A)^2 \cdot \left| \frac{\partial F}{\partial r} \right|^2 + \left| \frac{\partial F}{\partial \eta} \right|^2 \right). \tag{46}$$

Proof We have the following estimate from Lemma 5.2, which is independent of F:

$$|e_1|_g^{-2} \cdot \big|dF(e_1)\big|^2 = \big((-\log r^2 + A)^2 + G_1\big) \cdot \left| \frac{\partial F}{\partial \eta} \right|^2, \qquad |G_1| = O\big(r \cdot (-\log r^2 + A)\big).$$

From Lemma 5.3, there exists a positive constant, which is independent of F, such that the following holds:

$$\big|dF(e_2)\big|^2 \leq C \cdot \big(\big|\partial_r F\big|^2 + \big|\partial_\eta F\big|^2 \big).$$

Thus we have the following estimate from Lemma 5.4, which is independent of F:

$$|e_2|_g^{-2} \cdot \big|dF(e_2)\big|^2 = O\Big(r^2 \cdot (-\log r^2 + A)^2 \cdot \big(|\partial_r F|^2 + |\partial_\eta F|^2 \big) \Big).$$

We take the orthogonal frame $\tilde{f}_1, \ldots, \tilde{f}_{2l}$ of the orthogonal complement of $T\overline{\Delta}$, by applying the orthogonalization of Schmidt to $f_1, \ldots, f_{2l}$. Then it is easy to obtain the following estimate, which is independent of F:

$$|dF(\tilde{f}_i)|^2 = O\Big(\sum_i |\partial_i F|^2 + r^2 \cdot |\partial_r F|^2 + |\partial_\eta F|^2 \Big).$$

Then (46) immediately follows. ∎

5.1.4 An estimate for some twisted map

Let (E, ∇) be a flat connection on $X \times \overline{\Delta}^*$. Let h' be a C^∞-hermitian metric of $E_{|X\times\partial\Delta}$.

Lemma 5.9 *We have the unique continuous hermitian metric h of E satisfying the following:*

- *The restriction $h_{|P\times\overline{\Delta}^*}$ is tame pure imaginary harmonic metric of $(E,\nabla)_{|P\times\overline{\Delta}^*}$. Here the metric of $P\times\overline{\Delta}^*$ is given by $|z|^{-2}\cdot(-\log|z|^2+A)^{-2}\cdot dz\cdot d\bar{z}$.*
- $h_{|X\times\partial\overline{\Delta}} = h'$.

Proof See the subsubsection 4.2.1. ∎

For any point P, let φ^P denote the monodromy of the harmonic bundle $(E,h,\nabla)_{|P\times\overline{\Delta}^*}$. The conjugacy classes are independent of a choice of P. Thus we denote φ^P simply by φ. Let Ψ_h denote the twisted map corresponding to h.

Lemma 5.10 *There exists a positive constant C_0, which is independent of P, such that the following holds:*

$$\left|\frac{\partial\Psi_h}{\partial r}\right|^2\cdot r^2\cdot(-\log r^2+A)^2\leq C_0,\qquad \left|\left|\frac{\partial\Psi_h}{\partial\eta}\right|^2\cdot(-\log r^2+A)^2-\frac{\rho(\varphi)^2}{4\pi^2}\cdot(-\log r^2+A)^2\right|\leq C_0.$$

Proof It follows from Lemma 4.11. ∎

Lemma 5.11 *$\partial_i\Psi_h$ is defined almost everywhere, and we have the following, for any point $P\in X$:*

$$\bigl|\partial_i\Psi_h\bigr|^2(T,P)\leq\max\Bigl\{|\partial_i\Psi_h|^2(T,Q)\,\Bigm|\,Q\in\partial\overline{\Delta}\Bigr\}.$$

Proof It follows from Lemma 4.12. ∎

Proposition 5.1 *The following function is bounded on $X\times\overline{\Delta}^*$:*

$$\left|e(\Psi_h)-\frac{\rho(\varphi)^2}{4\pi^2}\cdot(-\log|z|^2+A)^2\right|.$$

Proof We obtain the boundedness of the following, due to Lemma 5.8, Lemma 5.10 and Lemma 5.11:

$$\left|e(\Psi_h)-\left|\frac{\partial\Psi_h}{\partial\eta}\right|^2\cdot(-\log|z|^2+A)^2\right|.$$

Then the desired boundedness immediately follows from Lemma 5.10. ∎

Corollary 5.1 *We have the boundedness of the following function on $X\times\overline{\Delta}^*$:*

$$\left|e(\Psi_h)-\frac{\rho(\varphi)^2}{4\pi^2}\bigl(-\log|\zeta|^2+b\bigr)^2\right|. \tag{47}$$

In particular, we obtain the following finiteness:

$$\int_{X\times\overline{\Delta}^*}\left|e(\Psi_h)-\frac{\rho(\varphi)^2}{4\pi^2}\bigl(-\log|\zeta|^2+b\bigr)^2\right|\cdot\mathrm{dvol}_g<\infty. \tag{48}$$

Proof The function $\bigl|\bigl(-\log|\zeta|^2+b\bigr)^2-\bigl(-\log|z|^2+A\bigr)^2\bigr|$ is bounded on $X\times\overline{\Delta}$, due to (43). Thus we obtain the boundedness (47). The finiteness (48) immediately follows from (47). ∎

5.1.5 A priori lower estimate of the energy for arbitrary map

Let b_1 be a C^∞-function on $X \times \overline{\Delta}$. The functions w, s and ϕ are given by the relations $e^{-(b+b_1)/2} \cdot \zeta = w = s \cdot e^{\sqrt{-1}\phi}$. We have $dw/w = d\zeta/\zeta - d(b+b_1)/2$. We use the coordinate change on $X \times \overline{\Delta}^*$ given as follows:

$$(x_1, \ldots, x_{2l}, r, \eta) \longleftrightarrow (x_1, \ldots, x_{2l}, s, \phi).$$

We have the equality $-\log|\zeta|^2 + b = -\log s^2 - b_1 > 0$.

A 1-form $\sum_i A_i \cdot dx_i + B \cdot ds + C \cdot d\phi$ on $X \times \overline{\Delta}^*$ is called independent of ϕ, if A_i, B and C are independent of ϕ. It is similarly defined that an i-form is independent of ϕ.

Lemma 5.12 *Let ω be a C^∞-form on $X \times \overline{\Delta}$. Then we have a decomposition $\omega = \omega_1 + \omega_2$, where ω_1 is independent of ϕ and $|\omega_2|_{g_1} = O(s)$.*

Proof We can check it elementarily by using Taylor development. ∎

Lemma 5.13 *We have the following estimate with respect to the Kahler form g_1 on $X \times \overline{\Delta}$:*

$$d\eta - d\phi = O(1), \qquad \frac{dr}{r} - \frac{ds}{s} = O(1).$$

Proof The first claim follows from the following:

$$2\sqrt{-1}d\eta = \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} = \frac{dw}{w} - \frac{d\bar{w}}{\bar{w}} + O(1) = 2\sqrt{-1}d\phi + O(1).$$

The second claim can be shown similarly. ∎

Lemma 5.14 *We have the following estimate of the volume form:*

$$\mathrm{dvol}_g = (H_1 + H_2) \cdot dx_1 \cdots dx_{2l} \cdot \frac{ds \cdot d\phi}{s \cdot (-\log s^2 - b_1)^2}.$$

Here H_1 is independent of ϕ and $0 < C_1 < H_1 < C_2$ for some positive constants C_i. We also have the estimate $|H_2| = O\big((-\log s^2 - b_1)^{-2}\big)$.

Proof We apply Lemma 5.12 to the C^∞-form $\partial\overline{\partial}b$, and then we have $g(\partial_i, \partial_j) = g_1(\partial_i, \partial_j) + K_1 \cdot \big(-\log s^2 - b_1\big)^{-1} + K_2$, where K_1 is independent of ϕ and $K_2 = O\big((-\log s^2 - b_1)^{-2}\big)$. Then the claim can be shown by an argument similar to the proof of Lemma 5.7 using Lemma 5.13. ∎

Lemma 5.15 *We have the following estimate:*

$$\frac{\partial\eta}{\partial\phi} = 1 + O(s), \qquad \frac{\partial r}{\partial\phi} = O(s^2).$$

Proof It follows from Lemma 5.13. ∎

Lemma 5.16 *We have the following estimate:*

$$|\partial_\phi|_g^2 = (1 + I_1) \cdot (-\log s^2 - b_1)^{-2}, \quad |I_1| = O\big(s \cdot (-\log s^2 - b_1)\big).$$

Proof We obtain the following from Lemma 5.15 and Lemma 5.2:

$$|\partial_\phi|_g^2 = \big(1 + O(s)\big) \cdot |\partial_\eta|_g^2 + O(s^4) \cdot \big|\partial_r\big|_g^2 + O(s^2) \cdot g\big(\partial_\eta, \partial_r\big) = (1 + I_1) \cdot (-\log s^2 - b_1)^2.$$

Thus we are done. ∎

Let $F : X \times \overline{\Delta}^* \longrightarrow \mathcal{PH}(r)/\langle\varphi\rangle$ be any twisted map. Then we obtain the following:

$$
\begin{aligned}
e_g(F)\cdot \mathrm{dvol}_g &\geq \bigl|\partial_\phi\bigr|_g^{-2}\cdot\bigl|\partial_\phi F\bigr|^2\cdot \mathrm{dvol}_g\\
&= (1+I_1)\cdot(-\log s^2-b_1)^2\cdot\bigl|\partial_\phi F\bigr|^2\times(H_1+H_2)\cdot dx_1\cdots dx_{2l}\cdot\frac{ds\cdot d\phi}{s\cdot(-\log s^2-b_1)^2}\\
&=: (H_1+H_3)\cdot\bigl|\partial_\phi F\bigr|^2\cdot dx_1\cdots dx_{2l}\cdot\frac{ds\cdot d\phi}{s}. \qquad (49)
\end{aligned}
$$

Here H_1 and H_2 are as in Lemma 5.14, and we put $H_3 := (1+I_1)\cdot(H_1+H_2)-H_1$. We remark $|H_3| = O\bigl((-\log s^2-b_1)^{-2}\bigr)$.

Lemma 5.17 *There is a function H_4, which is independent of F, satisfying $|H_4| = O\bigl((-\log s^2-b_1)^{-2}\bigr)$ and the following inequality:*

$$
\int_0^{2\pi}\bigl|\partial_\phi F\bigr|^2\cdot(H_1+H_3)\cdot s^{-1}\cdot d\phi \geq \int_0^{2\pi}\Bigl(\frac{\rho(\varphi)^2}{4\pi^2}-H_4\Bigr)\cdot(H_1+H_2)\cdot s^{-1}\cdot d\phi.
$$

Proof Due to Lemma 2.15, we have the following:

$$
\int_0^{2\pi}\bigl|\partial_\phi F\bigr|^2\cdot\bigl(1+H_1^{-1}\cdot H_3\bigr)\cdot d\phi \geq \rho(\varphi)^2\cdot\Bigl(\int_0^{2\pi}(1+H_1^{-1}\cdot H_3)^{-1}\cdot d\phi\Bigr)^{-1}.
$$

Since we have $|H_1^{-1}\cdot H_3| = O\bigl((-\log s^2-b_1)^{-2}\bigr)$, we obtain the following, for some H_4' such that $|H_4'| = O\bigl((-\log s^2-b_1)^{-2}\bigr)$:

$$
\int_0^{2\pi}\bigl|\partial_\phi F\bigr|^2\cdot\bigl(1+H_1^{-1}\cdot H_3\bigr)\cdot d\phi \geq \frac{\rho(\varphi)^2}{2\pi}-H_4'.
$$

Hence we obtain the following:

$$
\begin{aligned}
\int_0^{2\pi}\bigl|\partial_\phi F\bigr|^2\cdot(H_1+H_3)\cdot s^{-1}\cdot d\phi &\geq \int_0^{2\pi}\Bigl(\frac{\rho(\varphi)^2}{4\pi^2}-\frac{H_4'}{2\pi}\Bigr)\cdot H_1\cdot s^{-1}\cdot d\phi\\
&\geq \int_0^{2\pi}\Bigl(\frac{\rho(\varphi)^2}{4\pi^2}-H_4\Bigr)\cdot(H_1+H_2)\cdot s^{-1}\cdot d\phi. \qquad (50)
\end{aligned}
$$

Here we take H_4 appropriately by using $|H_2| = O\bigl((-\log s^2-b_1)^{-2}\bigr)$. Thus we are done. ∎

Let us consider the region $\mathcal{S}$ of the following form:

$$
\mathcal{S} := \bigl\{(P,Q)\in X\times\overline{\Delta}^*\bigm| R_1\leq -\log s(Q)\leq R_2\bigr\}.
$$

Corollary 5.2 *There is a function H_4, which is independent of F, satisfying $|H_4| = O\bigl((-\log|\zeta|^2+b)^{-2}\bigr)$ and the following inequality:*

$$
\int_{\mathcal{S}} e(F)\cdot\mathrm{dvol}_g \geq \int_{\mathcal{S}}\bigl|\partial_\phi F\bigr|^2\cdot\bigl|\partial_\phi\bigr|_g^{-2}\cdot\mathrm{dvol}_g \geq \int_{\mathcal{S}}\Bigl(\frac{\rho(\varphi)^2}{4\pi^2}-H_4\Bigr)\cdot(-\log|\zeta|^2+b)^2\cdot\mathrm{dvol}_g\,.
$$

Note the finiteness $\int_{X\times\overline{\Delta}^*}|H_4|\cdot(-\log|\zeta|^2+b)^2\,\mathrm{dvol}_g<\infty$. ∎

5.2 Around the intersection

5.2.1 Preliminary

We use the real coordinate (x_1,y_1,x_2,y_2) given by $z_i = \exp(\sqrt{-1}x_i - y_i)$ on $(\overline{\Delta}^*)^2$. For the moment, we use the Poincaré metric g_0 on $(\overline{\Delta}^*)^2$:

$$
g_0 := \sum_{a=1,2}\frac{dx_a\cdot dx_a+dy_a\cdot dy_a}{\bigl(2y_a+A\bigr)^2}.
$$

Let (E, ∇) be a flat bundle on $(\overline{\Delta}^*)^2$. Let f_i $(i = 1, 2)$ be monodromies around $z_i = 0$. We put as follows:

$$(\overline{\Delta}^*)^2 \supset Y := \bigl\{(z_1, z_2) \in (\overline{\Delta}^*)^2 \bigm| |z_1| = |z_2|\bigr\} \simeq]0, 1] \times T^2.$$

We have the twisted map $\psi' : Y \longrightarrow \mathcal{PH}(r)/\langle f_1, f_2\rangle$ given as follows:

$$\psi'(x_1, x_2, y_1, y_2) = F\bigl(B \cdot \log(y_1 + y_2 + A), x_1, x_2\bigr) = F\bigl(B \cdot \log(2y_1 + A), x_1, x_2\bigr).$$

(See the subsubsections 2.4.2 and 2.4.4.) Here B denotes sufficiently large positive constant. We remark that the morphism Φ in the subsubsection 2.4.4 is just twisted by the isomorphisms.

Let $\pi : (\overline{\Delta}^*_w)^2 \longrightarrow (\overline{\Delta}^*_z)^2$ be the map given by $\pi(w_1, w_2) = (w_1, |w_1| \cdot w_2)$. Then we obtain the isomorphism $\overline{\Delta}^*_{w_1} \times \partial\Delta_{w_2} \simeq Y$.

We use the real coordinate $(\xi_1, \xi_2, \eta_1, \eta_2)$ of $(\overline{\Delta}^*_w)^2$, given by $w_i = \exp\bigl(\sqrt{-1}\xi_i - \eta_i\bigr)$. Then we have the relation $x_i = \xi_i$ $(i = 1, 2)$, $y_1 = \eta_1$ and $y_2 = \eta_1 + \eta_2$.

We have the C^∞-metric h' of $\pi^{-1}(E)_{|\overline{\Delta}^* \times \partial\overline{\Delta}}$, which corresponds to a twisted map ψ'. We take the unique continuous metric h of $\pi^{-1}(E)$ on $(\overline{\Delta}^*)^2$ satisfying the following conditions:

- $h_{|\overline{\Delta}^* \times \partial\overline{\Delta}} = h'$.
- The restrictions $(\pi^{-1}(E), \pi^{-1}\nabla, h)_{|P \times \overline{\Delta}^*_{w_2}}$ is tame pure imaginary harmonic bundle, for any point $P \in \overline{\Delta}^*$.

Let $\Psi_h : (\overline{\Delta}^*)^2 \longrightarrow \mathcal{PH}(r)/\langle f_1, f_2\rangle$ denote the corresponding twisted map.

We have the following estimate, due to Lemma 2.19:

$$\left| \left|\frac{\partial\psi'}{\partial\xi_1}\right|^2 - \frac{\rho(\varphi)^2}{4\pi^2} \right| \leq C_1 \cdot e^{-C_2 \cdot B \log(2\eta_1 + A)} \leq \frac{C_3}{(2\eta_1 + A)^{C_2 B}}.$$

Hence we obtain the following, due to Lemma 4.12:

$$\left|\frac{\partial\Psi_h}{\partial\xi_1}\right|^2 \leq \frac{\rho(\varphi)^2}{4\pi^2} + \frac{C_3}{(2\eta_1 + A)^{C_2 B}}. \tag{51}$$

We have the following estimate, due to Lemma 2.20:

$$\left|\frac{\partial\psi'}{\partial\eta_1}\right| = \frac{B \cdot \sqrt{\sum \alpha_i^2}}{2\eta_1 + A}.$$

Therefore we obtain the following, due to Lemma 4.12:

$$\left|\frac{\partial\Psi_h}{\partial\eta_1}\right| \leq \frac{B \cdot \sqrt{\sum \alpha_i^2}}{2\eta_1 + A}. \tag{52}$$

Lemma 5.18 *There exists a positive constant C such that the following holds:*

$$\left|\frac{\partial\Psi_h}{\partial\eta_2}\right|^2 \leq \frac{C}{(2\eta_2 + A)^2}, \qquad \left| \left|\frac{\partial\Psi_h}{\partial\xi_2}\right|^2 - \frac{\rho(f_2)^2}{4\pi^2} \right| \leq \frac{C}{(2\eta_2 + A)^2}.$$

Proof It follows from Lemma 4.15. We remark that the boundary value $\tilde{\Phi}$ in the subsubsection 4.2.2 is obtained from Φ (see the subsubsection 2.4.4), and Φ is just twisted by isomorphisms from F, which is the boundary value we consider here. Hence we can use Lemma 4.15 here. ∎

5.2.2 The twisted maps and the estimates of their energy on Z_1 and Z_2

We reformulate the result in the subsubsection 5.2.1. We put $Z_1 := \big\{(z_1,z_2)\,\big|\,|z_1|\geq |z_2|\big\}$. We have the naturally defined projection $\pi_1 : Z_1 \longrightarrow \overline{\Delta}^*_{z_1}$. For any point $P\in\overline{\Delta}^*_{z_1}$, we have $\pi_1^{-1}(P)\simeq\big\{z_2\,\big|\,0<|z_2|\leq |z_1(P)|\big\}$. We take the continuous hermitian metric h_{Z_1} of $E_{|Z_1}$ satisfying the following:

- $h_{Z_1\,|\,Y}=h'$.
- For any point $P\in\overline{\Delta}^*_{z_1}$, the restriction $h_{Z_1\,|\,\pi_1^{-1}(P)}$ is a tame pure imaginary harmonic metric.

Lemma 5.19 *We have the following estimate:*

$$\left|\frac{\partial\Psi_{h_{Z_1}}}{\partial x_1}\right|^2\leq\frac{\rho(f_1)^2}{4\pi^2}+O\big((2y_1+A)^{-C_2B}\big),$$

$$\left|\frac{\partial\Psi_{h_{Z_1}}}{\partial x_2}\right|^2\leq\frac{\rho(f_2)^2}{4\pi^2}+O\big(\big(2(y_2-y_1)+A\big)^{-2}\big),$$

$$\left|\frac{\partial\Psi_{h_{Z_1}}}{\partial y_1}\right|^2=O\Big((2y_1+A)^{-2}+\big(2(y_1-y_2)+A\big)^{-2}\Big),$$

$$\left|\frac{\partial\Psi_{h_{Z_1}}}{\partial y_2}\right|^2=O\big(\big(2(y_2-y_1)+A\big)^{-2}\big).$$

Proof It follows from (51), (52), Lemma 5.18 and the relations $x_i=\xi_i$ $(i=1,2)$, $y_1=\eta_1$ and $y_2=\eta_1+\eta_2$. ∎

Lemma 5.20 *We have the following estimate on Z_1:*

$$|\partial_{z_1}|_{g_0}^{-2}\cdot\big|\partial_{z_1}\Psi_{h_{Z_1}}\big|^2\leq\frac{(2y_1+A)^2}{2}\cdot\left(\frac{\rho(f_1)^2}{4\pi^2}+O\Big((2y_1+A)^{-C_2B}+(2y_1+A)^{-2}+\big(2(y_2-y_1)+A\big)^{-2}\Big)\right).$$

$$|\partial_{z_2}|_{g_0}^{-2}\cdot\big|\partial_{z_2}\Psi_{h_{Z_1}}\big|^2\leq\frac{(2y_2+A)^2}{2}\left(\frac{\rho(f_2)^2}{4\pi^2}+O\Big(\big(2(y_2-y_1)+A\big)^{-2}\Big)\right).$$

Proof It follows from Lemma 5.19. ∎

Corollary 5.3 *We have the following estimate of the energy $e_{g_0}\big(\Psi_{h_{Z_1}}\big)$ with respect to the metric g_0 on Z_1:*

$$e_{g_0}\big(\Psi_{h_{Z_1}}\big)\leq\sum_{i=1,2}\frac{\rho(f_i)^2}{4\pi^2}(2y_i+A)^2+O\left(1+\frac{(2y_1+A)^2}{\big(2(y_2-y_1)+A\big)^2}+\frac{(2y_2+A)^2}{\big(2(y_2-y_1)+A\big)^2}\right).$$

The last term in the right hand side is integrable on Z_1 with respect to the measure dvol_{g_0} *induced by the metric g_0.* ∎

Similarly we put $Z_2 := \big\{(z_1,z_2)\,\big|\,|z_1|\geq|z_2|\big\}$. Let $\pi_2 : Z_2\longrightarrow\Delta^*_{z_2}$ be the naturally defined projection. We have the continuous hermitian metric h_{Z_2} of $E_{|Z_2}$ satisfying the following:

- $h_{Z_2\,|\,Y}=h'$.
- For any point $P\in\overline{\Delta}^*_{z_2}$, the restriction $h_{Z_2\,|\,\pi_2^{-1}(P)}$ is tame pure imaginary harmonic metric.

We obtain the following lemma.

Lemma 5.21 *Lemma* 5.19*, Lemma* 5.20 *and Corollary* 5.3 *for h_{Z_2} hold, when (x_1,y_1) and (x_2,y_2) are exchanged and Z_1 is replaced with Z_2.* ∎

Since h_{Z_1} and h_{Z_2} coincides on Y, they give the continuous and locally L_1^2 hermitian metric h on $(\overline{\Delta}^*)^2$.

Lemma 5.22 *There exists an integrable function J_i $(i=1,2)$ with respect to the measure* dvol_{g_0}*, such that the following holds for h:*

$$|\partial_{z_1}|_{g_0}^{-2}\cdot|\partial_{z_1}\Psi_h|^2=\frac{(2y_1+A)^2}{2}\cdot\frac{\rho(f_1)^2}{4\pi^2}+J_1$$

$$|\partial_{z_2}|_{g_0}^{-2}\cdot|\partial_{z_2}\Psi_h|^2=\frac{(2y_2+A)^2}{2}\frac{\rho(f_2)^2}{4\pi^2}+J_2$$

$$e_{g_0}\big(\Psi_h\big)\leq\sum_{i=1,2}\frac{\rho(f_i)^2}{4\pi^2}(-2y_i+A)^2+2(J_1+J_2).$$

Proof It follows from Corollary 5.3 and Lemma 5.21. ∎

5.2.3 Perturbation of the metric and the estimate of the energy function

Let g_1 be a C^∞-Kahler metric of $\overline{\Delta}^2$. Let us consider the Kahler metric $g=g_0+g_1$. We put as follows:

$$e_1:=\partial_{z_1},\quad e_2:=\partial_{z_2}-\frac{g(\partial_{z_2},\partial_{z_1})}{|\partial_{z_1}|_g^2}\cdot\partial_{z_1}.$$

We have the following estimate:

$$\big|e_2\big|_g^2=\big|\partial_{z_2}\big|_g^2\cdot\left(1-\frac{\big|g(\partial_{z_1},\partial_{z_2})\big|^2}{\big|\partial_{z_1}\big|^2}\right)=\big|\partial_{z_2}\big|_g^2\cdot\Big(1+O\Big(|z_1|^2\cdot|z_2|^2\cdot\big(-\log|z_1|^2+A\big)^2\cdot\big(-\log|z_2|^2+A\big)^2\Big)\Big).\tag{53}$$

Lemma 5.23 *Let $\Phi:(\overline{\Delta}^*)^2\longrightarrow\mathcal{PH}(r)/\langle f_1,f_2\rangle$ be any twisted map. The energy function $e_g(\Phi)$ can be described as follows:*

$$e_g\big(\Phi\big)=2\sum_{i=1,2}\big(1+G_i\big)\cdot\big|\partial_{z_1}\big|_g^{-2}\cdot\big|\partial_{z_i}\Phi\big|^2.$$

We have the estimate of G_i:

$$|G_i|=O\Big(|z_1|\cdot|z_2|\cdot\big(-\log|z_1|^2+A\big)\cdot\big(-\log|z_2|^2+A\big)\Big).$$

The estimate is independent of a choice of Φ.

Proof We have the following equality:

$$\big|d\Phi(e_2)\big|^2=\big|\partial_{z_2}\Phi\big|^2-2\,\mathrm{Re}\left(\frac{g(\partial_{z_1},\partial_{z_2})}{\big|\partial_{z_1}\big|_g^2}\big(\partial_{z_2}\Phi,\partial_{z_1}\Phi\big)\right)+\frac{\big|g(\partial_{z_1},\partial_{z_2})\big|^2}{\big|\partial_{z_1}\big|_g^4}\cdot\big|\partial_{z_1}\Phi\big|^2.$$

Hence we have the following inequality:

$$\Big|\big|d\Phi(e_2)\big|^2-\big|\partial_{z_2}\Phi\big|^2\Big|\leq2\cdot\frac{\big|g(\partial_{z_1},\partial_{z_1})\big|}{\big|\partial_{z_1}\big|_g^2}\cdot\big|\partial_{z_1}\Phi\big|\cdot\big|\partial_{z_2}\Phi\big|+\frac{\big|g(\partial_{z_1},\partial_{z_2})\big|^2}{\big|\partial_{z_1}\big|_g^4}\cdot\big|\partial_{z_1}\Phi\big|^2.$$

Hence we obtain the following inequality:

$$\begin{aligned}\Big||e_2|_g^{-2}\cdot\big|d\Phi(e_2)\big|^2-\big|\partial_{z_2}\big|_g^{-2}\cdot\big|\partial_{z_2}\Phi\big|^2\Big|&\leq|e_2|_g^{-2}\cdot\Big|\big|d\Phi(e_2)\big|^2-\big|\partial_{z_2}\Phi\big|^2\Big|+\Big||e_2|_g^{-2}-\big|\partial_{z_2}\big|_g^{-2}\Big|\cdot\big|\partial_{z_2}\Phi\big|^2\\&\leq C_1\frac{\big|g(\partial_{z_1},\partial_{z_2})\big|}{\big|\partial_{z_1}\big|_g^2\cdot\big|\partial_{z_2}\big|_g^2}\cdot\big|\partial_{z_2}\Phi\big|\cdot\big|\partial_{z_1}\Phi\big|+C_1\frac{\big|g(\partial_{z_1},\partial_{z_2})\big|^2}{\big|\partial_{z_1}\big|_g^2\cdot\big|\partial_{z_2}\big|_g^2}\cdot\big|\partial_{z_1}\big|_g^{-2}\cdot\big|\partial_{z_1}\Phi\big|^2+\Big||e_2|_g^{-2}-\big|\partial_{z_2}\big|_g^{-2}\Big|\cdot\big|\partial_{z_2}\Phi\big|^2.\end{aligned}\tag{54}$$

Here C_1 is a positive constant, which depends only on g_1. The first term in the right hand side of (54) is dominated as follows:

$$C_1 \cdot \frac{\big|g(\partial_{z_1},\partial_{z_2})\big|}{\big|\partial_{z_1}\big|_g\cdot\big|\partial_{z_2}\big|_g}\cdot\Big(\big|\partial_{z_1}\big|_g^{-1}\cdot\big|\partial_{z_1}\Phi\big|\Big)\cdot\Big(\big|\partial_{z_2}\big|_g^{-1}\cdot\big|\partial_{z_2}\Phi\big|\Big)\leq C_1\cdot\frac{\big|g(\partial_{z_1},\partial_{z_2})\big|}{\big|\partial_{z_1}\big|_g\cdot\big|\partial_{z_2}\big|_g}\cdot\Big(\big|\partial_{z_1}\big|_g^{-2}\cdot\big|\Phi\big|^2+\big|\partial_{z_2}\big|_g^{-2}\cdot\big|\Phi\big|^2\Big).$$

We can control the third term in the right hand side of (54) by using (53). Then it is easy to derive the claim of the lemma, by using the formula $e_g\big(\Phi\big)=2\sum_{i=1,2}|e_i|_g^{-2}\cdot\big|d\Phi(e_i)\big|^2$. ∎

Lemma 5.24 *Let Ψ_h be the twisted map given in the subsubsection* 5.2.2. *We have the following estimate:*

$$\big|e_g(\Psi_h)-e_{g_0}(\Psi_h)\big|=O\left(\sum_{i=1,2}\big(-\log|z_i|^2+A\big)\cdot|z_i|\cdot\big|\partial_{z_i}\big|_{g_0}^{-2}\cdot\big|\partial_{z_i}\Psi_h\big|^2\right). \tag{55}$$

The right hand side of (55) *is integrable with respect to the measure* dvol_{g_0}.

Proof The estimate (55) follows from Lemma 5.23 and the equality $\big|\partial_{z_i}\big|_g^2=\big|\partial_{z_i}\big|_{g_0}^2+\big|\partial_{z_i}\big|_{g_1}^2$. The integrability of the right hand side of (55) follows from Lemma 5.22. ∎

5.2.4 The volume form and the estimate of the energy

The volume form for the metric g is given as follows:

$$\mathrm{dvol}_g=\Big||\partial_{z_1}|_g^2\cdot|\partial_{z_2}|_g^2-g(\partial_{z_1},\partial_{z_2})\Big|\cdot dz_1d\bar{z}_1dz_2d\bar{z}_2.$$

We have $|\partial_{z_i}|_g^2=|z_i|^{-2}(-\log|z_i|^2+A)^2+g_1(\partial_{z_i},\partial_{z_i})$, and $g(\partial_{z_1},\partial_{z_2})=g_1(\partial_{z_1},\partial_{z_2})$. Here $g_1(\partial_{z_i},\partial_{z_j})$ is C^∞ on $\overline{\Delta}^2$.

Lemma 5.25 dvol_g *is of the following form:*

$$\mathrm{dvol}_g=\Big((1+F_1)\cdot(1+F_2)+F_3\Big)\cdot\mathrm{dvol}_{g_0}\,.$$

Here F_i $(i=1,2)$ are of the form $\tilde{F}_i\cdot|z_i|^2\cdot(-\log|z_i|^2+A)^2$ for C^∞-functions $\tilde{F}_i$ on $\overline{\Delta}^2$, and F_3 is of the form $\tilde{F}_3\cdot\prod_{i=1,2}|z_i|^2\cdot(-\log|z_i|^2+A)^2$ for C^∞-function $\tilde{F}_3$. ∎

Lemma 5.26 *There exists an integrable function J_3 with respect to* dvol_g *such that the following holds:*

$$e_g\big(\Psi_h\big)\leq\sum_{i=1,2}\frac{\rho(f_i)^2}{4\pi^2}\big(-\log|z_i|^2+A\big)^2+J_3. \tag{56}$$

Proof From Lemma 5.22 and Lemma 5.24, there exists an integrable function J_3 with respect to dvol_{g_0} such that the estimate (56) holds. The integrability of J_3 with respect to dvol_g follows from Lemma 5.25. ∎

Corollary 5.4 *There exists an integrable function J_3 with respect to* dvol_g, *such that the following holds for any compact region K of $(\overline{\Delta}^*)^2$:*

$$\int_K e_g(\Psi_h)\,\mathrm{dvol}_g\leq\int_K\left(\frac{\rho(f_1)^2}{4\pi^2}(-\log|z_1|^2+A)^2+\frac{\rho(f_2)^2}{4\pi^2}(-\log|z_2|^2+A)^2+J_3\right)\cdot\mathrm{dvol}_g\,.$$

∎

5.2.5 A priori lower estimate for arbitrary map

Let $\Phi : (\overline{\Delta}^*)^2 \longrightarrow \mathcal{PH}(r)\big/\langle f_1, f_2\rangle$ be any twisted map. We would like to obtain the lower bound of the energy with respect to the metric g given in the subsubsection 5.2.3.

Lemma 5.27 *We have the following:*

$$\mathrm{dvol}_g = (1+F_3')(1+F_1)(1+F_2)\,\mathrm{dvol}_{g_0}\,.$$

Here F_1 and F_2 are as in Lemma 5.25, and we have $|F_3'| = O\big(|z_1|\cdot|z_2|\big)$.

Proof It immediately follows from Lemma 5.25. ∎

We use the real coordinate $z_i = \exp\big(\sqrt{-1}x_i - y_i\big)$ as before. Recall Lemma 5.23. We have the following:

$$\begin{aligned}2\big(1+G_i\big)\cdot\big|\partial_{z_i}\big|_g^{-2}\cdot\big|\partial_{z_i}\Phi\big|^2 &= \big(1+G_i\big)\cdot\Big(\big|\partial_{x_i}\big|_g^{-2}\cdot\big|\partial_{x_i}\Phi\big|^2+\big|\partial_{y_i}\big|_g^{-2}\cdot\big|\partial_{y_i}\Phi\big|^2\Big)\\ &\geq \big(1+G_i\big)\cdot\Big(\big|\partial_{x_i}\big|_g^{-2}\cdot\big|\partial_{x_i}\Phi\big|^2\Big) = (1+H_i)\cdot\big|\partial_{x_i}\Phi\big|^2\cdot\big(-\log|z_i|^2+A\big)^2. \qquad (57)\end{aligned}$$

Here we have the estimate $\big|H_i\big| = O\big(|z_i|^{1/2}\big)$. The estimate is independent of a choice of Φ.

Let us consider the case $i=1$. We can decompose $(1+F_2) = (1+F_2'')\cdot(1+F_2')$ such that F_2' is independent of z_2, and the estimate $|F_2''| = O(|z_1|)$ holds. (See Lemma 5.25.) We put $I_1 := (1+H_1)\cdot(1+F_3')\cdot(1+F_1)-1$, and then we have the estimate $|I_1| = O(|z_1|^{1/2})$. The estimate is independent of a choice of Φ. Then we have the following:

$$(1+G_1)\cdot\big|\partial_{x_1}\Phi\big|^2\cdot\big|\partial_{x_1}\big|_g^{-2}\cdot\mathrm{dvol}_g \;=\Big((1+I_1)\cdot\big|\partial_{x_1}\Phi\big|^2\cdot dy_1\cdot dx_1\Big)\cdot(1+F_2')\cdot\frac{dy_2\cdot dx_2}{(2y_2+A)^2} \qquad (58)$$

Lemma 5.28 *There is a function J_5 satisfying $|J_5| = O(|z_1|^{1/2})$ and the following inequality:*

$$\int_0^{2\pi}(1+I_1)\cdot\big|\partial_{x_1}\Phi\big|^2\cdot dx_1 \geq \int_0^{2\pi}\Big(\frac{\rho(f_1)^2}{4\pi^2}-J_5\Big)\cdot dx_1.$$

Proof It follows from Lemma 2.15. See the proof of Lemma 5.17. ∎

We have the natural projection of $p : (\overline{\Delta}^*)^2 \longrightarrow]0,1]^2$. For any compact subset K of $]0,1]^2$, we put $\tilde{K} = p^{-1}(K)$.

Lemma 5.29 *There exists an integrable function J_6 with respect to the measure* dvol_g *such that the following holds for any compact subset $K \subset]0,1]^2$:*

$$\int_{\tilde{K}}(1+I_1)\cdot\big|\partial_{x_1}\Phi\big|^2\cdot dx_1\cdot dy_1\cdot(1+F_2')\cdot\frac{dx_2\cdot dy_2}{(2y_2+A)^2} \geq \int_{\tilde{K}}\Big(\frac{\rho(f_1)^2}{4\pi^2}\cdot\big(-\log|z_1|^2+A\big)^2-J_6\Big)\cdot\mathrm{dvol}_g\,.$$

In other words, we have the following inequality:

$$\begin{aligned}\int_{\tilde{K}}2\cdot(1+G_1)\cdot\big|\partial_{z_1}\Phi\big|^2\cdot\big|\partial_{z_1}\big|_g^{-2}\cdot\mathrm{dvol}_g &\geq \int_{\tilde{K}}(1+G_1)\cdot\big|\partial_{x_1}\Phi\big|^2\cdot\big|\partial_{x_1}\big|_g^{-2}\cdot\mathrm{dvol}_g\\ &\geq \int_{\tilde{K}}\Big(\frac{\rho(f_1)^2}{4\pi^2}\cdot\big(-\log|z_1|^2+A\big)^2-J_6\Big)\cdot\mathrm{dvol}_g\,. \qquad (59)\end{aligned}$$

Proof We have only to put $J_6 := J_5\cdot(1+F_3')^{-1}\cdot(1+F_1)^{-1}\cdot(1+F_2'')^{-1}\cdot(-\log|z_1|+A)^2$. ∎

Similarly, there exists an integrable function J_7 with respect to the measure dvol_g such that the following holds for any compact subset $K \subset]0,1]^2$:

$$\begin{aligned}\int_{\tilde{K}}2\cdot(1+G_2)\cdot\big|\partial_{z_2}\Phi\big|^2\cdot\big|\partial_{z_2}\big|_g^{-2}\cdot\mathrm{dvol}_g &\geq \int_{\tilde{K}}(1+G_2)\cdot\big|\partial_{x_2}\Phi\big|^2\cdot\big|\partial_{x_2}\big|_g^{-2}\cdot\mathrm{dvol}_g\\ &\geq \int_{\tilde{K}}\Big(\frac{\rho(f_2)^2}{4\pi^2}\cdot\big(-\log|z_2|^2+A\big)^2-J_7\Big)\cdot\mathrm{dvol}_g\,. \qquad (60)\end{aligned}$$

Corollary 5.5 *There exists an integrable function* J_8 *with respect to* dvol_g *such that the following inequality holds for any twisted map* Φ *and for any compact subset* $K \subset]0,1]$*:*

$$\int_{\tilde{K}} e(\Phi)\,\mathrm{dvol}_g \geq \int_{\tilde{K}} \left(\sum_{i=1,2} \frac{\rho(f_i)^2}{4\pi^2} \cdot (-\log|z_i|^2 + A)^2 - J_8 \right) \cdot \mathrm{dvol}_g\,.$$

Proof It follows from (59), (60) and Lemma 5.23. ∎

5.3 On $X - D$

5.3.1 hermitian metrics of line bundles and neighbourhoods of divisors

Let X be a compact complex surface. Let D_i $(i = 1, \ldots, l)$ be a normal crossing divisor of X such that $D = \bigcup D_i$ is normal crossing. We have the canonical section $s_i : \mathcal{O} \longrightarrow \mathcal{O}(D_i)$.

Let P be a point of $D_i \cap D_j$. We take a sufficiently small neighbourhood U_P of P. We may assume $U_P \cap D_k = \emptyset$ unless $k = i, j$. We may also assume that $U_P \cap D_i$ and $U_P \cap D_j$ are holomorphically isomorphic to an open disc Δ. We may assume that we can take a holomorphic trivialization e_i and e_j of $\mathcal{O}(D_i)$ and $\mathcal{O}(D_j)$ respectively. The holomorphic functions z_i and z_j are determined by $s_i = z_i \cdot e_i$ and $s_j = z_j \cdot e_j$. Then we may assume that $\varphi_P = (z_i, z_j) : U_P \longrightarrow \boldsymbol{C}^2$ gives a holomorphic embedding. We may also assume that $\varphi_P(U_P) \supset \overline{\Delta}^2$.

We have the hermitian metrics $h_{i\,U_P}$ (resp. $h_{j\,U_P}$) of $\mathcal{O}(D_i)_{|U_P}$ (resp. $\mathcal{O}(D_j)_{|U_P}$) given by $|e_i| = 1$ (resp. $|e_j| = 1$). By shrinking U_P appropriately, we can take a C^∞-hermitian metric h_i of L_i such that $h_{i\,|\,U_P} = h_{i\,U_P}$ for any $P \in D_i \cap D_j$.

We have the metric $dz_i \cdot d\bar{z}_i + dz_j \cdot d\bar{z}_j$ on U_P. Take a hermitian metric of the tangent bundle TX, which is not necessarily Kahler, such that the following holds:

- $g_{U_P} = dz_i \cdot d\bar{z}_i + dz_j \cdot d\bar{z}_j$. We shrink U_P if it is necessary.
- The hermitian metric g induces the orthogonal decomposition $TX_{|D_i} = TD_i \oplus N_{D_i}(X)$. We have the natural isomorphism $N_{D_i}(X) \simeq \mathcal{O}(D_i)_{|D_i}$. Then the restriction of g to $N_{D_i}(X)$ is same as the restriction of h_i to $\mathcal{O}(D_i)_{|D_i}$.

We have the exponential map $T(X)_{|D_i} \longrightarrow X$. Let us consider the restriction $\exp_i : N_{D_i}(X) \longrightarrow X$. For any $R \in \boldsymbol{R}_{>0}$, we put as follows:

$$N'_{i\,R} := \bigl\{ v \in N_{D_i}(X) \,\big|\, h_i(v,v) < R^2 \bigr\}, \qquad N_{i\,R} := \exp_i\Bigl(N'_{i\,R}\Bigr) \subset X.$$

It is well known that there exists a positive number R_0 such that $N'_{i\,R}$ and $N_{i\,R}$ are diffeomorphic for any $R \leq R_0$.

We have the naturally defined projection $\pi_i : N'_{i\,R} \longrightarrow D_i$. In the case $R \leq R_0$, it induces the projection $\pi_i : N_{i\,R} \longrightarrow D_i$. For any point $P \in D_i$ and for any $R \leq R_0$, we put $N'_{i\,R\,|\,P} := \pi_i^{-1}(P)$ and $N_{i\,R\,|\,P} = \exp_i\bigl(N'_{i\,R\,|\,P}\bigr)$. We have the natural complex structure of $N'_{i\,R\,|\,P} \simeq \Delta(R)$, which induces the complex structure of $N_{i\,R\,|\,P}$. The inclusion $N_{i\,R\,|\,P} \longrightarrow X$ is not necessarily holomorphic embedding. However the following lemma is obtained from our construction. (Compare the lemma with the condition 5.1.)

Lemma 5.30 *The inclusion of the tangent spaces* $T_{(P,O)}N_{i\,R\,|\,P} \longrightarrow T_{(P,O)}X$ *is compatible with their complex structures.* ∎

Let P be a point of $D_i \cap D_j$. The holomorphic function z_j gives a holomorphic coordinate of $D_i \cap U_P$. The bundle $N_{D_i}(X)_{U_P \cap D_i}$ is trivialized by e_i.

Lemma 5.31 *There exists a positive constant* R_1 *such that the map* $\exp_i : N'_{i\,R} \longrightarrow X$ *is given by* $(z_j, z_i \cdot e_i) \longmapsto (z_i, z_j)$*, for any* $R \leq R_1$*.*

Proof Since the metric on U_P is given by $dz_i \cdot d\bar{z}_i + dz_j \cdot d\bar{z}_j$, the claim is clear. ∎

Lemma 5.32 *There exists a positive constant R_2 such that the following holds, for any $R \le R_2$:*

- *The set of the connected components of $N_{i\,R} \cap N_{j\,R}$ corresponds bijectively to $D_i \cap D_j$.*
- *Let $(N_{i\,R} \cap N_{j\,R})_P$ denote the connected components of $N_{i\,R} \cap N_{j\,R}$ corresponding to P. Then $(N_{i\,R} \cap N_{j\,R})_P \subset U_P$.*
- *$\varphi_P\big((N_{i\,R} \cap N_{j\,R})_P\big) = \Delta(R)^2$.*

Proof It is clear from our construction. ∎

For a positive constant C, we put $\tilde{h}_i = C^{-2} \cdot h_i$, $\tilde{e}_i = C \cdot e_i$ and $\tilde{z}_i = C^{-1} \cdot z_i$. If C is taken appropriately, we may assume that R_0, R_1 and R_2 are larger than 1. Hence we may assume $R_i > 1$ from the beginning. In the following, we use the notation N_i and N_i' to denote $N_{i\,1}$ and $N_{i\,1}'$ respectively.

5.3.2 The Kahler metric and the decomposition

We assume X is a Kahler surface with the Kahler metric g_1.

Remark 5.2 *We do not assume that the Kahler metric g_1 is not necessarily same as the hermitian metric of TX used in the subsubsection* 5.3.1. ∎

Let ω_1 be the associated Kahler form. We take a positive constant A such that $|s_i|^2 < e^A$ for any i. If a positive constant C is sufficiently large, the following form ω also gives a Kahler form ([7]):

$$\omega = C \cdot \omega_1 - \sum_i \sqrt{-1}\partial\overline{\partial} \log\big(-\log|s_i|^2 + A\big).$$

We put $X^\circ = X - \bigcup_i N_i$. It is C^∞ compact submanifold of X, which possibly has a boundary with the corner.

We denote the closure of N_i by $\overline{N}_i$. It is C^∞-submanifold of X with the boundary. We put $M_P := \overline{N}_i \cap \overline{N}_j$, which is holomorphically isomorphic to $\overline{\Delta}^2$. It is a compact submanifold of X with the boundary and the corner. We may assume that the norm of the canonical section s_k is constant on M_P unless $k = i, j$. We can identify s_k and z_k for $k = i, j$ on M_P. We would like to apply the result in the subsection 5.2.

We put $D_i^\circ := D_i - \bigcup_{j \neq i}(N_j \cap D_i)$, which is a C^∞ compact submanifold of D_i with the boundary. Let $\partial_P D_i$ denote the component of the boundary of D_i contained in U_P.

$\overline{N}_i$ can be regarded as $\overline{\Delta}$-bundle over D_i in the C^∞-category. We put $\overline{N}_i^\circ := D_i^\circ \times_{D_i} \overline{N}_i$. We also put $N_i^\circ := D_i^\circ \times_{D_i} N_i$. We put $\partial_0 \overline{N}_i^\circ := \overline{N}_i^\circ - N_i^\circ$. We put $\partial_P \overline{N}_i := \partial_P D_i^\circ \times_{D_i^\circ} \overline{N}_i^\circ$.

Let $D_i^\circ = \coprod U_{i\,j}$ be disjoint unions, where $U_{i\,j}$ denote subregions of D_i°, which are isomorphic to subregions of $\boldsymbol{C}$. We put $N_{i\,j}^\circ := N_i^\circ \times_{D_i^\circ} U_{i\,j}$. The fibration $N_{i\,j}^\circ \longrightarrow U_{i\,j}$ can be trivialized in the C^∞-category. We would like to apply the result in the subsection 5.1 to $N_{i\,j}$, by putting $-\log|\zeta|^2 - b = \log|s_i|^2 + A$ (the subsubsection 5.1.1) and $-\log s^2 = -\sum_{i=1}^{l}\big(-\log|s_i|^2 + A\big)$ (the subsubsection 5.1.5). We remark that the constant A here is different from the function A in the subsubsection 5.1.1. We also remark Lemma 5.30.

5.3.3 The construction of maps with the controlled energy

Let (E, ∇) be a flat bundle of rank r on $X - D$. Then we have a homotopy class of twisted maps $X - D \longrightarrow \mathcal{PH}(r)/\pi_1(X)$. We take any C^∞-twisted map $F^\circ : X^\circ \longrightarrow \mathcal{PH}(r)/\pi_1(X)$, which corresponds to a hermitian metric h° of $(E, \nabla)_{|X^\circ}$.

We take a continuous hermitian metric h_i° of $(E, \nabla)_{|\overline{N}_i^\circ - D_i^\circ}$, satisfying the following:

- The restrictions of h_i° and h° to $\partial_0 \overline{N}_i$ are same.
- The restriction of h_i° to $(E, \nabla)_{|N_{i\,|\,P}^*}$ is a tame pure imaginary harmonic metric. Here we put $N_{i\,|\,P}^* := N_{i\,|\,P} - \{(P, O)\}$, and the conformal structure is induced by that of $N_{i\,|\,P}'$.

(See the subsection 5.1.) The corresponding twisted map is denoted by F_i°.

Let P be any point of $D_i \cap D_j$. We have the continuous hermitian metric h_{M_P} of $(E,\nabla)_{|M_P\setminus D}$ as in the subsection 5.2. We may assume that the restrictions of h_{M_P} and h° to $\partial\overline{\Delta}\times\partial\overline{\Delta}$ are same, if we modify h° appropriately. Then we also obtain $h_{M_P\,|\,\partial_P\overline{N}_i^{\circ}} = h^{\circ}_{i\,|\,\partial_P\overline{N}_i^{\circ}}$ due to our construction.

Hence we obtain the continuous hermitian metric h_0 of (E,∇) on $X-D$, such that $h_{0\,|\,X^{\circ}} = h^{\circ}$, $h_{0\,|\,N_i^{\circ}} = h_i^{\circ}$ and $h_{0\,|\,M_P} = h_{M_P}$. Let Ψ_{h_0} denote the corresponding twisted map.

Let us take a small loop γ_i around D_i. Then we obtain the monodromy φ_i with respect to γ_i. It is easy to see that the number $\rho(\varphi_i)$ is independent of a choice of γ_i. We denote the number by ρ_i.

Let K be any compact subset of $X-D$.

Lemma 5.33 *There exist the integrable functions J_{10} on $X-D$ with respect to the measure* dvol_g *such that the following holds, for any compact subset $K\subset X-D$:*

$$\int_K e(\Psi_{h_0})\cdot \mathrm{dvol}_g \leq \int_K \left(\sum_i \frac{\rho_i^2}{4\pi^2}\cdot\big(-\log|s_i|^2+A\big)^2 + J_{10}\right)\cdot \mathrm{dvol}_g\,.$$

Proof It follows from our construction. See Corollary 5.1 and Corollary 5.4. ∎

For any real numbers R, R_1, R_2, we put as follows:

$$X(R) := \Big\{P\in X\,\Big|\,\sum -\log|s_i|(P)\leq R\Big\},\quad X(R_1,R_2) := \Big\{P\in X\,\Big|\,R_1\leq\sum -\log|s_i|(P)\leq R_2\Big\}.$$

When we consider $X(R)$ (resp. $X(R_1,R_2)$), the number R (resp. R_1 and R_2) is chosen such that the boundary of $X(R)$ (resp. $X(R_1,R_2)$) is C^{∞}.

We can take a C^{∞}-twisted map $F_N : X(N)\longrightarrow \mathcal{PH}(r)/\pi_1(X)$, which approximates $\Psi_{h_0\,|\,X(N)}$ in L_1^2-sense sufficiently closely, such that there exists an integrable function J on $X-N$ such that the following holds for any compact subset $K\subset X(N)$:

$$\int_K \big|e(\Psi_{h_0})-e(F_N)\big|\cdot \mathrm{dvol} < \int_K J\cdot \mathrm{dvol}\,.$$

5.3.4 A priori lower bound for energy of any map

Let P be a point of $D_i\cap D_j$. We may assume that $|s_k|$ is constant on M_P unless $k=i,j$.

Lemma 5.34 *There exists an integrable function J_{11} on $X-D$ with respect to the measure* dvol_g, *such that the following holds:*

- *For any twisted harmonic map $F : X(R_1,R_2)\longrightarrow \mathcal{PH}(r)/\pi_1(X)$, the following holds:*

$$\int_{X(R_1,R_2)} e(F)\cdot \mathrm{dvol}_g \geq \int_{X(R_1,R_2)}\left(\sum_i \frac{\rho_i^2}{4\pi^2}\big(-\log|s_i|^2+A\big)^2 - J_{11}\right)\cdot \mathrm{dvol}_g\,.$$

Proof It follows from Lemma 5.17 and Corollary 5.5. ∎

6 The existence of tame pure imaginary pluri-harmonic metric

6.1 Harmonic metric of a semisimple flat bundle on a quasi compact Kahler surface

6.1.1 The existence of harmonic metric

Let X be a compact Kahler surface. Let D be a normal crossing divisor of X. Let (E,∇) be a semisimple flat bundle on $X-D$. Let us see the existence of harmonic metric of (E,∇). We use a notation in the subsection 5.3.

Due to the theorem of Hamilton-Schoen-Corlette (see the proof of Theorem 2.1 of [3]), we can take a twisted harmonic map $\Psi_N : X(N) \longrightarrow \mathcal{PH}(r)/\pi_1(X)$ satisfying the following:

$$\Psi_{N\,|\,\partial X(N)} = F_{N\,|\,\partial X(N)}, \qquad \int_{X(N)} e(\Psi_N)\cdot \mathrm{dvol}_g \leq \int_{X(N)} e(F_N)\cdot \mathrm{dvol}_g\,.$$

In the case $N > k$, we have the following inequalities, due to Lemma 5.34:

$$\begin{aligned}\int_{X(N)} e(\Psi_N)\cdot \mathrm{dvol}_g &= \int_{X(k)} e(\Psi_N)\cdot \mathrm{dvol}_g + \int_{X(k,N)} e(\Psi_N)\cdot \mathrm{dvol}_g \\ &\geq \int_{X(k)} e(\Psi_N)\cdot \mathrm{dvol}_g + \int_{X(k,N)} \Big(\sum \frac{\rho_i^2}{4\pi^2}\cdot\big(-\log|s_i|^2 + A\big)^2 - J_{11}\Big)\cdot \mathrm{dvol}_g\,. \qquad (61)\end{aligned}$$

We have the following inequality due to Lemma 5.33:

$$\int_{X(k,N)} \Big(\sum \frac{\rho_i^2}{4\pi^2}\big(-\log|s_i|^2 + A\big)^2\Big)\cdot \mathrm{dvol}_g \geq \int_{X(k,N)} \big(e(\Psi_{h_0}) - J_{10}\big)\cdot \mathrm{dvol}_g$$

Due to our choice of F_N, we have $\int_{X(k,N)} e(\Psi_{h_0})\,\mathrm{dvol}_g \geq \int_{X(k,N)} (e(F_N) - J)\cdot \mathrm{dvol}_g$ for some integrable function J, which is independent of N. Thus we obtain the following inequality:

$$\int_{X(k,N)} \Big(\sum \frac{\rho_i^2}{4\pi^2}\big(-\log|s_i|^2 + A\big)\Big)\cdot \mathrm{dvol}_g \geq \int_{X(k,N)} (e(F_N) - J_{10} - J)\cdot \mathrm{dvol}_g$$

Then we obtain the following:

$$\int_{X(N)} e(F_N)\,\mathrm{dvol}_g \geq \int_{X(N)} e(\Psi_N)\cdot \mathrm{dvol}_g \geq \int_{X(k)} e(\Psi_N)\cdot \mathrm{dvol}_g + \int_{X(k,N)} e(F_N)\cdot \mathrm{dvol}_g - \int_{X(k,N)} (J_{10}+J_{11}+J)\cdot \mathrm{dvol}_g$$

Thus we obtain the following:

$$\begin{aligned}\int_{X(k)} e(\Psi_N)\cdot \mathrm{dvol}_g &\leq \int_{X(k)} e(F_N)\cdot \mathrm{dvol}_g + \int_{X(k,N)} (J_{10} + J_{11} + J)\cdot \mathrm{dvol}_g \\ &\leq \int_{X(k)} e(\Psi_{h_0})\cdot \mathrm{dvol}_g + \int_{X(k,N)} (J_{10} + J_{11} + 2J)\cdot \mathrm{dvol}_g, \\ &\leq \int_{X(k)} e(\Psi_{h_0})\cdot \mathrm{dvol}_g + \int_{X(k,\infty)} (J_{10} + J_{11} + 2J)\cdot \mathrm{dvol}_g\,. \qquad (62)\end{aligned}$$

Lemma 6.1 *Assume that (E,∇) is semisimple. Then there exists an infinite subset $\boldsymbol{n}_1$ of $\boldsymbol{N}$ such that the sequence $\big\{\Psi_N \,\big|\, N \in \boldsymbol{n}_1\big\}$ is C^∞-convergent to a twisted harmonic map $\Psi_\infty : X - D \longrightarrow \mathcal{PH}(r)/\pi_1(X)$.*

Proof Since we have the estimate of the energy (62), we have only to apply the arguments in the section 2 in Jost-Yau [21] (using semisimplicity) and [37]. ∎

Let h denote the harmonic metric corresponding to Ψ_∞, and θ and $\theta^\dagger$ denote the corresponding $(1,0)$-form and $(0,1)$-form respectively. We denote Ψ_∞ by Ψ_h.

Thus we obtain the harmonic metric h for any semisimple flat bundle (E,∇) on $X - D$. We will show that the harmonic metric h constructed is, in fact, pluri harmonic (Proposition 6.1) and tame pure imaginary (Theorem 6.1).

6.1.2 The decomposition and the energy

We put $N^{\circ}_{i\,j}(R_1,R_2) := N^{\circ}_{i\,j}\cap X(R_1,R_2)$ and $N^{\circ}_{i\,j}(R) := N^{\circ}_{i\,j}\cap X(R)$. Due to Corollary 5.2, there exist integrable functions $J_{i,j}$ on $N^{\circ}_{i\,j}\setminus D$ with respect to dvol_g, such that the following holds for any $R_1 < R_2$:

$$\begin{aligned}\int_{N^{\circ}_{i\,j}(R_1,R_2)} e(\Psi_h)\cdot \mathrm{dvol}_g &\geq \int_{N^{\circ}_{i\,j}(R_1,R_2)} \big|\partial_\phi\Psi_h\big|^2\cdot\big|\partial_\phi\big|_g^{-2}\cdot \mathrm{dvol}_g \\ &\geq \int_{N^{\circ}_{i\,j}(R_1,R_2)} \Big(\frac{\rho_i^2}{4\pi^2}\cdot\big(-\log|s_i|^2 + A\big)^2 - J_{i,j}\Big)\cdot \mathrm{dvol}_g\,. \qquad (63)\end{aligned}$$

We put $M_P(R_1,R_2) := M_P \cap X(R_1,R_2)$ and $M_P(R) := M_P \cap X(R)$. Due to Corollary 5.5, there exists an integrable function J_P on $M_P \setminus D$ on dvol_g, such that the following inequality for any $R_1 < R_2$:

$$\int_{M_P(R_1,R_2)} e\bigl(\Psi_h\bigr)\cdot \mathrm{dvol}_g \geq \int_{M_P(R_1,R_2)} \Bigl(\sum_i \frac{\rho_i^2}{4\pi^2}\cdot\bigl(-\log|s_i|^2+A\bigr)^2 - J_P\Bigr)\cdot \mathrm{dvol}_g\,. \tag{64}$$

On the other hand, we obtain the following inequality due to our construction (see (62) and Lemma 5.33):

$$\int_{X(R)} e(\Psi_h)\cdot \mathrm{dvol}_g \leq \int_{X(R)} \Bigl(\sum_i \frac{\rho_i^2}{4\pi^2}\cdot\bigl(-\log|s_i|^2+A\bigr)^2 + J_{10}\Bigr)\cdot \mathrm{dvol}_g + \int_{X(R,\infty)} (J_{10}+J_{11}+2J)\cdot \mathrm{dvol}_g\,.$$

The second term in the right hand side converges to 0 when $R\to\infty$. Hence there exists a positive constant $\tilde{C}$, such that the following holds for any R:

$$\int_{X(R)} e(\Psi_h)\cdot \mathrm{dvol}_g \leq \int_{X(R)} \sum_i \frac{\rho_i^2}{4\pi^2}\cdot\bigl(-\log|s_i|^2+A\bigr)^2\cdot \mathrm{dvol}_g + \tilde{C}. \tag{65}$$

From (63), (64) and (65), there exist positive constants $C_{i\,j}$, such that the following holds for any $R>0$:

$$\int_{N^\circ_{i\,j}(R)} e(\Psi_h)\cdot \mathrm{dvol}_g \leq \int_{N^\circ_{i\,j}(R)} \frac{\rho_i^2}{4\pi^2}\cdot\bigl(-\log|s_i|^2+A\bigr)^2\cdot \mathrm{dvol}_g + C_{i\,j}. \tag{66}$$

Similarly there exist constants C_P for any $P\in D_i\cap D_j$, such that the following holds for any $R>0$:

$$\int_{M^\circ_P(R)} e(\Psi_h)\cdot \mathrm{dvol}_g \leq \int_{M^\circ_P(R)} \Bigl(\frac{\rho_i^2}{4\pi^2}\cdot\bigl(-\log|s_i|^2+A\bigr)^2 + \frac{\rho_j^2}{4\pi^2}\cdot\bigl(-\log|s_j|^2+A\bigr)^2\Bigr)\cdot \mathrm{dvol}_g + C_P. \tag{67}$$

6.1.3 Estimate of the energy on $N^\circ_{i\,j}\setminus D$ and $N^\circ_i\setminus D$

We use the coordinate as in the subsubsection 5.1.5.

Lemma 6.2 *The function $e(\Psi_h) - \bigl|\partial_\phi\Psi_h\bigr|^2\cdot\bigl|\partial_\phi\bigr|^{-2}$ is integrable.*
There exists a positive constant $C_{i\,j}$ such that the following inequality holds for any $R_1<R_2$:

$$\int_{N^\circ_{i\,j}(R_1,R_2)} \bigl|\partial_\phi\Psi_h\bigr|^2\cdot\bigl|\partial_\phi\bigr|^{-2}\cdot \mathrm{dvol}_g \leq \int_{N^\circ_{i\,j}(R_1,R_2)} \frac{\rho_i^2}{4\pi^2}\cdot\bigl(-\log|s_i|^2+b\bigr)^2\cdot \mathrm{dvol}_g + C_{i\,j}. \tag{68}$$

Proof The first claim follows from (63), (66) and the positivity $e(\Psi_h) - \bigl|\partial_\phi\Psi_h\bigr|^2\cdot\bigl|\partial_\phi\bigr|^{-2}$. The second claim follows from (63) and (66). ∎

Corollary 6.1 $\bigl|\partial_\phi\Psi_h\bigr|^2$ *is integrable.*

Proof It is easy to derive the integrability from (68) and the estimate $\bigl|\partial_\phi\bigr|^2 \sim \bigl(-\log|s_i|^2+b\bigr)^{-2}$. ∎

Lemma 6.3 $\bigl|\partial_s\Psi_h\bigr|^2\cdot\bigl|\partial_s\bigr|^{-2}$ *is integrable.*

Proof We put $e_2' := \partial_s - g(\partial_s,\partial_\phi)\cdot|\partial_\phi|^{-2}\cdot\partial_\phi$. Then we obtain $\bigl|\partial_s\Psi_h\bigr|^2 \leq C\cdot\bigl(\bigl|d\Psi_h(e_2')\bigr|^2 + \bigl|\partial_\phi\Psi_h\bigr|^2\bigr)$. The function $\bigl|d\Psi_h(e_2')\bigr|^2\cdot\bigl|e_2'\bigr|^{-2}$ is integrable due to Lemma 6.2. Then it is easy to check the claim. (Use the argument in the subsubsection 5.1.3). ∎

Let $\theta=\theta_1+\theta_2$ be the orthogonal decomposition such that θ_1 is of the following form:

$$\theta_1 = \frac{1}{4}H^{-1}\cdot dH\bigl(\partial_\phi - \sqrt{-1}J\cdot\partial_\phi\bigr)\cdot\bigl(d\phi - \sqrt{-1}J\cdot d\phi\bigr).$$

Here J denotes the complex structure of X. Then we have the following:

$$\bigl|\theta_1\bigr|^2 = \frac{1}{16}\bigl|d\Psi_h\bigl(\partial_\phi - \sqrt{-1}J\partial_\phi\bigr)\bigr|^2\cdot 2\cdot\bigl|d\phi\bigr|^2 = \frac{1}{8}\Bigl(\bigl|d\Psi_h\bigl(\partial_\phi\bigr)\bigr|^2\cdot\bigl|\partial_\phi\bigr|^{-2} + \bigl|d\Psi_h\bigl(J\partial_\phi\bigr)\bigr|^2\cdot\bigl|J\partial_\phi\bigr|^{-2}\Bigr).$$

Lemma 6.4 *We have the following estimate:*

$$J\partial_\phi = -s\partial_s + O(s^2)\cdot\partial_s + O(s)\cdot\partial_\phi + O(s)\partial_{x_i}.$$

Proof Since we have $dw/w - d\zeta/\zeta = O(1)$, we have the following with respect to the Kahler metric g_1 on X:

$$J\cdot d\phi = \frac{ds}{s} + O(1), \quad J\cdot\frac{ds}{s} = -d\phi + O(1).$$

Hence we have $s^{-1}ds\bigl(J\partial_\phi\bigr) = (J\cdot s^{-1}ds)(\partial_\phi) = -1 + O(s)$ and $d\phi\bigl(J\partial_\phi\bigr) = \bigl(J\cdot d\phi\bigr)(\partial_\phi) = O(s)$. We also have $J\cdot dx_i = O(1)$ with respect to g_1. Then the claim follows. ∎

Lemma 6.5 $\bigl|d\Psi_h\bigl(J\partial_\phi\bigr)\bigr|^2\cdot\bigl|J\partial_\phi\bigr|^{-2}$ *is integrable.*

Proof From Lemma 6.4, we obtain $d\Psi_h\bigl(J\partial_\phi\bigr) = O(s)\cdot\partial_s\Psi_h + O(s)\cdot\partial_\phi\Psi_h + O(s)\cdot\partial_{x_i}\Psi_h$, and thus we have the following estimate:

$$\begin{aligned}\bigl|d\Psi_h(J\partial_\phi)\bigr|^2\cdot\bigl|J\partial_\phi\bigr|^{-2} &= O\Bigl(s^2\cdot\bigl|\partial_\phi\Psi_h\bigr|^2\cdot\bigl|\partial_\phi\bigr|^{-2}\Bigr) + O\Bigl(\bigl|\partial_s\Psi_h\bigr|^2\cdot\bigl|\partial_s\bigr|^{-2}\Bigr) + O\Bigl(s^2\cdot(-\log|s|^2)^2\cdot\sum_i\bigl|\partial_{x_i}\Psi_h\bigr|^2\Bigr)\\ &= O\Bigl(s^2\cdot\bigl|\partial_\phi\Psi_h\bigr|^2\cdot\bigl|\partial_\phi\bigr|^{-2}\Bigr) + O\Bigl(\bigl|\partial_s\Psi_h\bigr|^2\cdot\bigl|\partial_s\bigr|^{-2}\Bigr) + s\cdot O\Bigl(e\bigl(\Psi_h\bigr)\Bigr). \end{aligned}\tag{69}$$

Then it is easy to derive the integrability of $s\cdot e\bigl(\Psi_h\bigr)$ from from Lemma 6.2, and thus the integrability of $\bigl|d\Psi_h\bigl(J\partial_\phi\bigr)\bigr|^2\cdot\bigl|J\partial_\phi\bigr|^{-2}$ follows from Corollary 6.1 and Lemma 6.3. ∎

Lemma 6.6 *There are integrable functions $\breve{J}_{i\,j}$ and a positive constant $\breve{C}_{i\,j}$ such that the following holds, for any $R_1 < R_2$:*

$$\begin{aligned}\int_{N^\circ_{i\,j}(R_1,R_2)}\Bigl(\frac{\rho_i^2}{32\pi^2}\cdot\bigl(-\log|s_i|^2 + A\bigr)^2 - \breve{J}_{i\,j}\Bigr)\cdot\mathrm{dvol}_g &\leq \int_{N^\circ_{i\,j}(R_1,R_2)}\bigl|\theta_1\bigr|^2\cdot\mathrm{dvol}_g\\ &\leq \int_{N^\circ_{i\,j}(R_1,R_2)}\frac{\rho_i^2}{32\pi^2}\cdot\bigl(-\log|s_i|^2 + A\bigr)^2\cdot\mathrm{dvol}_g + \breve{C}_{i\,j}.\end{aligned}\tag{70}$$

We have the following finiteness on N_i°:

$$\int_{N_i^\circ - D_i^\circ}|\theta_1|^2\cdot\frac{\mathrm{dvol}_g}{\bigl(-\log|s_i|^2 + A\bigr)^2} < \infty.\tag{71}$$

The function $\bigl|\theta_2\bigr|^2$ is integrable.

Proof The estimate for $\bigl|\theta_1\bigr|^2$ follows from the estimate of $\bigl|\partial_\phi\Psi_h\bigr|\cdot\bigl|\partial_\phi\bigr|^{-2}$ and the integrability of $\bigl|d\Psi_h(J\partial_\phi)\bigr|\cdot\bigl|J\partial_\phi\bigr|^{-2}$. The integrability of $\bigl|\theta_2\bigr|^2$ follows from the estimates of $\bigl|\theta_1\bigr|^2$ and $e(\Psi_h)$. ∎

Corollary 6.2 *We have the finiteness:*

$$\int_{N_i^\circ - D_i^\circ}\bigl|\theta_1\bigr|\cdot\bigl|\theta_2\bigr|\frac{\mathrm{dvol}_g}{(-\log|s_i|^2 + A)} < \infty.$$

Proof It follows from the L^2-property of θ_2 and $\theta_1\cdot(-\log|s_i|^2 + A)^{-1}$. ∎

6.1.4 Estimate of the energy on $M_P \setminus D$

Let P be a point of $D_i \cap D_j$. For simplicity, we consider the case $(i,j) = (1,2)$. Note we have $s_i = z_i$ $(i = 1,2)$ on M_P. We use the result in the subsubsection 5.2.5.

From the inequalities (59), (60), (67) and Lemma 5.23, there exist constants C_i such that the following inequalities hold, for any $R_1 < R_2$:

$$\int_{M_P^{\circ}(R_1,R_2)} \big(1+G_i\big) \cdot \big|\partial_{x_i}\Psi_h\big|^2 \cdot \big|\partial_{x_i}\big|_g^{-2} \cdot \mathrm{dvol}_g \leq \int_{M_P^{\circ}(R_1,R_2)} 2 \cdot \big(1+G_i\big) \cdot \big|\partial_{z_i}\Psi_h\big|^2 \cdot \big|\partial_{z_i}\big|_g^{-2} \cdot \mathrm{dvol}_g$$
$$\leq \int_{M_P^{\circ}(R_1,R_2)} \frac{\rho_i^2}{4\pi^2} \cdot \big(-\log|z_i|^2 + A\big)^2 \cdot \mathrm{dvol}_g + C_i. \tag{72}$$

Here G_i are given in Lemma 5.23, and we have the estimate $G_i = O\big(|z_1|\cdot|z_2|\cdot(-\log|z_1|^2+A)\cdot(-\log|z_2|^2+A)\big)$.

Lemma 6.7 *The functions* $\big|\partial_{y_i}\Psi_h\big| \cdot \big|\partial_{y_i}\big|^{-2}$ $(i=1,2)$ *are integrable with respect to* dvol_g.

Proof The integrability of $(1+G_i)\cdot\big|\partial_{y_i}\Psi_h\big|\cdot\big|\partial_{y_i}\big|_g^{-2}$ follows from (59), (60) (72) and the relation $2\cdot\big|\partial_{z_i}\big|^{-2}\cdot\big|\partial_{z_i}\Psi_h\big|^2 = \big|\partial_{x_i}\big|^{-2}\cdot\big|\partial_{x_i}\Psi_h\big|^2 + \big|\partial_{y_i}\big|^{-2}\cdot\big|\partial_{y_i}\Psi_h\big|^2$. Then it is easy to derive the lemma. ∎

Recall the argument in the subsubsection 5.2.5. We put $M_{P\,1} := \big\{(y_1,z_2) \in \boldsymbol{R}_{\geq 0} \times \overline{\Delta}^*\big\}$. We have the measure $d\mu_0 := (1+F_2')\cdot(2y_2+A)^{-2}\cdot dy_1\cdot dx_2\cdot dy_2$. Due to Lemma 5.28, there exists a function J_{100} on $M_{P\,1}$ with the estimate $|J_{100}| = O(e^{-y_1/2})$, such that the following holds:

$$\int_0^{2\pi} \big(1+I_1\big)\cdot\big|\partial_{x_1}\Psi_h\big|^2\cdot dx_1 \geq \frac{\rho(f_1)^2}{2\pi} - J_{100}.$$

We put $M_{P\,1}(R_1,R_2) := \big\{(y_1,z_2)\in M_{P\,1}\,\big|\,R_1 \leq y_1+y_2 \leq R_2\big\}$. From (72), we obtain the following inequality:

$$\int_{M_{P\,1}(R_1,R_2)} J_{101}\cdot d\mu \leq \int_{M_{P\,1}(R_1,R_2)} J_{100}\cdot d\mu + C_1 \leq \tilde{C}_1.$$

Here C_1 and $\tilde{C}_1$ are positive constant, which are independent of $R_1 < R_2$. Then we obtain the integrability of J_{101} with respect to $d\mu$.

We put as follows:

$$J_{102} := \int_0^{2\pi} \big|\partial_{x_1}\Psi_h\big|^2\cdot dx_1 - \frac{\rho(f_1)^2}{2\pi}, \quad J_{103} := \int_0^{2\pi}\big|I_1\big|\cdot\big|\partial_{x_1}\Psi_h\big|^2\cdot dx_1. \tag{73}$$

From the estimate $|I_1| = O\big(e^{-y_1/2}\big)$ and the integrability of J_{101}, we obtain the integrability of J_{103}. Hence we also obtain the integrability of J_{102}.

Let us consider the following function on $\boldsymbol{R}^2_{\geq 0}$:

$$\Lambda_i := \int_0^{2\pi}\int_0^{2\pi}\big|\partial_{x_i}\Psi_h\big|^2\cdot dx_1\cdot dx_2\cdot\big(2y_i+A\big)^2 - \rho(f_i)^2\cdot\big(2y_i+A\big)^2.$$

Lemma 6.8 *The functions* Λ_i $(i=1,2)$ *are integrable with respect to* $(2y_1+A)^{-2}\cdot(2y_2+A)^{-2}\cdot dy_1\cdot dy_2$.

Proof The case $i=1$ follows from the integrability of J_{102} above. The case $i=2$ can be discussed similarly. ∎

We decompose $\theta = \theta_1 + \theta_2$, where θ_i are of the form $f_i\cdot dz_i/z_i = f_i\cdot(\sqrt{-1}dx_i - dy_i)$. We have the following equality:

$$\big|f_i\big|_h^2 = \frac{1}{16}\cdot\Big(\big|\partial_{x_i}\Psi_h\big|^2 + \big|\partial_{y_i}\Psi_h\big|^2\Big).$$

Let us consider the following functions on $\boldsymbol{R}^2_{\geq 0}$:

$$\Phi_i := \int_0^{2\pi}\int_0^{2\pi}\big|f_i\big|_h^2\cdot dx_1\cdot dx_2\cdot(2y_i+A)^2.$$

Lemma 6.9 *We have the decomposition* $\Phi_i = (16)^{-1} \cdot \rho(f_i)^2 \cdot (2y_i + A)^2 + J_{4\,i}$, *where* $J_{4\,i}$ *are integrable with respect to the measure* $(2y_1 + A)^{-2} \cdot (2y_2 + A)^{-2} \cdot dy_1 \cdot dy_2$.

Proof It follows from Lemma 6.7 and Lemma 6.8. ▮

6.2 Preliminary integrability

6.2.1 Statement and some reductions

Lemma 6.10 $\overline{\partial}\theta$ *and* θ^2 *are* L^2 *with respect to the measure* dvol_g.

We will prove Lemma 6.10 in the next subsubsections. Let us take a function $\psi : \boldsymbol{R}_{\geq 0} \longrightarrow \boldsymbol{R}$ satisfying the following:

$$0 \leq \psi \leq 1, \qquad \psi(x) = 1 \quad (x \leq 1/2), \qquad \psi(x) = 0 \quad (x \geq 2/3).$$

For any positive number N, we put as follows:

$$\chi_N := \prod_i \psi\Bigl(-\frac{\log|s_i|^2}{N}\Bigr).$$

Lemma 6.11 *When* N *is sufficiently large, we have* $\chi_N = \psi\bigl(-N^{-1} \cdot \log|s_i|^2\bigr)$ *on* N_i°, *and we have* $\chi_N = \psi\bigl(-N^{-1} \cdot \log|s_i|^2\bigr) \cdot \psi\bigl(-N^{-1} \cdot \log|s_j|^2\bigr)$ *on* M_P *for* $P \in D_i \cap D_j$. ▮

Due to Proposition 2.1, we have the following equality:

$$\begin{aligned}\int_{X-D} \chi_N \cdot \Bigl(C_1 \cdot \bigl|[\theta,\theta]\bigr|_h^2 + C_2 \cdot \bigl|\overline{\partial}\theta\bigr|_h^2\Bigr) \cdot \omega^n &= \int_{X-D} \overline{\partial}\partial\chi_N \cdot \bigl\langle \theta,\, \theta \bigr\rangle \cdot \omega^{n-2} \\ &= \sum \int_{N_i^{\circ}} \overline{\partial}\partial\chi_N \cdot \bigl\langle \theta,\, \theta \bigr\rangle \cdot \omega^{n-2} + \sum_P \int_{M_P} \overline{\partial}\partial\chi_N \cdot \bigl\langle \theta,\, \theta \bigr\rangle \cdot \omega^{n-2}. \qquad (74)\end{aligned}$$

Since the integrand of the left hand side is positive, we have only to show that each term in the right hand side is bounded independently of N. We will check such boundedness in the subsubsections 6.2.2 and 6.2.3.

6.2.2 On $N_{i,j}^{\circ} \setminus D$ and $N_i^{\circ} \setminus D$

We use the results in the subsubsection 6.1.3. On $N_{i,j}^{\circ}$, we have the following equality:

$$\partial\overline{\partial}\chi_N = \frac{1}{N^2} \cdot \psi''\Bigl(-\frac{\log|s_i|^2}{N}\Bigr) \cdot \partial\log|s_i|^2 \wedge \overline{\partial}\log|s_i|^2 - \frac{1}{N}\psi'\Bigl(\frac{-\log|s_i|^2}{N}\Bigr) \cdot \partial\overline{\partial}\log|s_i|^2.$$

We put $\tau = -\partial\overline{\partial}\log|s_i|^2$. It is a C^{∞}-closed form on X, and it gives the first Chern class of $\mathcal{O}(D_i)$ in the cohomology level. We put as follows:

$$\mu = J \cdot d\phi + \sqrt{-1}d\phi, \qquad G_1 := \partial\log|s_i|^2 - \mu.$$

Then we have $\bigl|G_1\bigr| = O(1)$ with respect to the Kahler form g_1 of X.

Let $\theta = \theta_1 + \theta_2$ be the orthogonal decomposition as in the subsubsection 6.1.3. Recall that θ_1 is of the form $f \cdot \mu$. We have the following:

$$\begin{aligned}&\bigl\langle \theta,\theta \bigr\rangle \cdot \partial\log|s_i|^2 \cdot \overline{\partial}\log|s_i|^2 \\ &= \bigl\langle \theta_1,\theta_1 \bigr\rangle \cdot G_1 \cdot \bar{G}_1 + \bigl\langle \theta_1,\theta_2 \bigr\rangle \cdot G_1 \cdot \bigl(\bar{\mu} + \bar{G}_1\bigr) + \bigl\langle \theta_2,\theta_1 \bigr\rangle \cdot (\mu + G_1) \cdot \bar{G}_1 + \bigl\langle \theta_2,\theta_2 \bigr\rangle \cdot (\mu + G_1) \cdot \bigl(\bar{\mu} + \bar{G}_1\bigr). \qquad (75)\end{aligned}$$

It is easy to check the following estimates:

$$\bigl\langle \theta_1,\theta_1 \bigr\rangle \cdot G_1 \cdot \bar{G}_1 \cdot \bigl(-\log|s_i|^2 + A\bigr)^{-2} = O\Bigl(\bigl|\theta_1\bigr|^2 \cdot \bigl(-\log|s_i|^2 + A\bigr)^{-2} \cdot \mathrm{dvol}_g\Bigr),$$

$$\langle \theta_1,\theta_2\rangle \cdot G_1\cdot (\bar{\mu}+\bar{G}_1)\cdot \bigl(-\log|s_i|^2+A\bigr)^{-2}=O\Bigl(|\theta_1|\cdot \bigl(-\log|s_i|^2+A\bigr)^{-1}\cdot |\theta_2|\cdot \mathrm{dvol}_g\Bigr),$$

$$\langle \theta_2,\theta_1\rangle \cdot (\mu+G_1)\cdot \bar{G}_1\cdot \bigl(-\log|s_i|^2+A\bigr)^{-2}=O\Bigl(|\theta_2|\cdot|\theta_1|\cdot \bigl(-\log|s_i|^2+A\bigr)^{-1}\cdot \mathrm{dvol}_g\Bigr),$$

$$\langle \theta_2,\theta_2\rangle \cdot (\mu+G_1)\cdot (\bar{\mu}+\bar{G}_1)\cdot \bigl(-\log|s_i|^2+A\bigr)^{-2}=O\Bigl(|\theta_2|^2\cdot \mathrm{dvol}_g\Bigr).$$

The right hand sides are integrable due to Lemma 6.6 and Corollary 6.2. We have the boundedness:

$$\frac{\bigl(-\log|s_i|^2+A\bigr)^2}{N^2}\cdot \psi''\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr)\leq C.$$

Since the support of the function $\psi''\bigl(-N^{-1}\cdot\log|s_i|^2\bigr)$ goes to infinity when $N\to\infty$, we obtain the following convergence:

$$\lim_{N\to\infty}\int_{N_i^{\circ}}\langle\theta,\theta\rangle\cdot\psi''\left(\frac{-1}{N}\log|s_i|^2\right)\cdot\frac{1}{N^2}\cdot\partial\log|s_i|^2\wedge\overline{\partial}\log|s_i|^2=0. \tag{76}$$

We decompose as $\langle\theta,\theta\rangle=\langle\theta_1,\theta_1\rangle+\langle\theta_1,\theta_2\rangle+\langle\theta_2,\theta_1\rangle+\langle\theta_2,\theta_2\rangle$. It is easy to check the following estimates:

$$\langle \theta_1,\theta_2\rangle\cdot\tau\cdot\bigl(-\log|s_i|^2+A\bigr)^{-1}=-\overline{\langle\theta_2,\theta_1\rangle}\cdot\tau\cdot\bigl(-\log|s_i|^2+A\bigr)^{-1}=O\Bigl(|\theta_1|\cdot|\theta_2|\cdot\bigl(-\log|s_i|^2+A\bigr)^{-1}\cdot\mathrm{dvol}_g\Bigr),$$

$$\langle\theta_2,\theta_2\rangle\cdot\tau\cdot\bigl(-\log|s_i|^2+A\bigr)^{-1}=O\Bigl(|\theta_2|^2\cdot\mathrm{dvol}_g\cdot\bigl(-\log|s_i|^2+A\bigr)^{-1}\Bigr).$$

The right hand sides are integrable due to Lemma 6.6 and Corollary 6.2. We have the boundedness of the functions $-N^{-1}\cdot\log|s_i|^2\cdot\psi'\bigl(-N^{-1}\cdot\log|s_i|^2\bigr)$, independently of N, and the supports of the functions go to infinity. Hence we obtain the following convergence:

$$\lim_{N\to\infty}\int_{N_i^{\circ}}\Bigl(\langle\theta_2,\theta_1\rangle+\langle\theta_1,\theta_2\rangle+\langle\theta_2,\theta_2\rangle\Bigr)\cdot\frac{1}{N}\cdot\psi'\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr)\wedge\tau=0. \tag{77}$$

To estimate the remained term, we use the coordinate as in the subsubsection 5.1.5. We put $\tau_0:=\tau_{|D_i^{\circ}}$. Let $\pi_i:N_i^{\circ}\longrightarrow D_i^{\circ}$ be the projection. We have the decomposition $\tau=\pi_i^*\tau_0+\tau_1$. Then we have the estimate $|\tau_1|=O(s)$. Recall that θ_1 is of the form $f\cdot\mu$. Then we have the following:

$$\begin{aligned}\langle\theta_1,\theta_1\rangle\wedge\tau&=\langle\theta_1,\theta_1\rangle\wedge\pi_i^*\tau_0+\langle\theta_1,\theta_1\rangle\wedge\tau_1\\&=-2\sqrt{-1}\cdot|f|^2\cdot\frac{ds\cdot d\phi}{s}\wedge\pi_i^*\tau_0+O\Bigl(|f|^2\cdot ds\cdot d\phi\wedge\pi_i^*\tau_0+|\theta_1|^2\cdot s^2\cdot\mathrm{dvol}_g\Bigr).\end{aligned} \tag{78}$$

Since $|\theta_1|^2\cdot\bigl(-\log s-b_1\bigr)^{-2}\cdot\mathrm{dvol}_g$ is integrable due to Lemma 6.6, the last term does not contribute to the limit when $N\to\infty$. We have the equality:

$$|f|^2=\frac{1}{16}\Bigl(\bigl|d\Psi_h(\partial_\phi)\bigr|^2+\bigl|d\Psi_h(J\partial_\phi)\bigr|^2\Bigr)\cdot(1+O(s)).$$

We have only to consider $|\partial_\phi\Psi_h|^2/16$ instead of $|f|^2$, due to the integrability of the other terms.

Let us consider the following integral:

$$\int_{N_{i\,j}^{\circ}\setminus D}\frac{1}{N}\psi'\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr)\cdot|\partial_\phi\Psi_h|^2\cdot\frac{ds\cdot d\phi}{s}\wedge\pi_i^*\tau_0=\int_{W_{i\,j}\times]0,1]}\frac{1}{N}\psi'\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr)\cdot\Phi\frac{ds}{s}\wedge\pi^*\tau_0.$$

Here we put as follows:

$$\Phi:=\int_0^{2\pi}|\partial_\phi\Psi_h|^2\cdot d\phi.$$

Due to Corollary 5.2, there exists an integrable function J_{25} on $W_{i\,j}\times]0,1]$ with respect to $s^{-1}ds\cdot\mathrm{dvol}_{W_{i\,j}}$, such that the following holds:

$$\frac{\rho_i^2}{2\pi}-J_{25}\leq\Phi.$$

There exists a positive constant C such that the following holds, for any $0 < a < b < 1$:

$$\int_{W_{i\,j}\times[a,b]} \Phi \cdot \frac{ds}{s} \cdot \mathrm{dvol}_{W_{i\,j}} \leq \int_{W_{i\,j}\times[a,b]} \Bigl(\frac{\rho_i^2}{2\pi}\Bigr) \cdot \frac{ds}{s} \cdot \mathrm{dvol}_{W_{i\,j}} + C. \tag{79}$$

We put $J_{27} := \Phi - (2\pi)^{-1} \cdot \rho_i^2 + J_{25}$, which is a positive function. It is easy to derive the integrability of J_{27} from (79). Namely, we have the decomposition $\Phi = \frac{\rho_i^2}{2\pi} + J_{28}$, where J_{28} is integrable with respect to $s^{-1} \cdot ds \wedge \mathrm{dvol}_{W_{i\,j}}$. We have the following:

$$\begin{aligned}
\int_{W_{i\,j}\times]0,1]} \frac{1}{N} \cdot \psi'\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr) \cdot \Phi \cdot \frac{ds}{s} \wedge \pi^*\tau_0 &= \int_{W_{i\,j}\times]0,1]} \frac{1}{N}\psi'\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr) \cdot \Bigl(\frac{\rho_i^2}{2\pi} + J_{28}\Bigr) \cdot \frac{ds}{s} \wedge \pi^*\tau_0 \\
&= \frac{\rho_i^2}{4\pi}\int_{W_{i\,j}} \tau_0 + \int_{W_{i\,j}\times]0,1]} \frac{1}{N}\psi'\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr) \cdot J_{28} \cdot \frac{ds}{s} \wedge \pi^*\tau_0. \qquad (80)
\end{aligned}$$

The second term converges to 0 when $N \to \infty$.

Let us consider the limit of the integrals over N_i°. Then we obtain the following convergence:

$$\lim_{N\to\infty} \int_{N_i^\circ} \bigl\langle \theta_1, \theta_1 \bigr\rangle \cdot \frac{1}{N} \cdot \psi'\Bigl(\frac{-1}{N}\log|s_i|^2\Bigr) \wedge \tau = C \cdot \rho_i^2 \cdot (D_i, D_i).$$

Here (D_i, D_i) denotes the self intersection number of D_i in X, and C denotes the constant which is independent of X, D, (E, ∇).

Thus we obtain the following convergence:

$$\lim_{N\to\infty} \int_{N_i^\circ} \bigl\langle \theta, \theta \bigr\rangle \wedge \partial\overline{\partial}\chi_N = C \cdot \rho_i^2 \bigl(D_i, D_i\bigr). \tag{81}$$

In particular, we obtain the uniform boundedness on the integrals over N_i°.

6.2.3 On $M_P \setminus D$

Let P be a point of $D_i \cap D_j$. For simplicity, we consider the case $(i,j) = (1,2)$. Note we have $s_i = z_i$ $(i = 1,2)$ on M_P. We have the following equality:

$$\begin{aligned}
\partial\overline{\partial}\chi_N = {} & \psi''\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi\bigl(-N^{-2}\log|z_2|^2\bigr) \cdot N^{-2} \cdot \frac{dz_1 \cdot d\bar{z}_1}{|z_1|^2} \\
& + \psi'\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi'\bigl(-N^{-2}\log|z_2|^2\bigr) \cdot N^{-2} \cdot \frac{dz_1 \cdot d\bar{z}_2}{z_1 \cdot \bar{z}_2} \\
& + \psi'\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi'\bigl(-N^{-2}\log|z_2|^2\bigr) \cdot N^{-2} \cdot \frac{dz_2 \cdot d\bar{z}_1}{z_2 \cdot \bar{z}_1} \\
& + \psi\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi''\bigl(-N^{-2}\log|z_2|^2\bigr) \cdot N^{-2} \cdot \frac{dz_2 \cdot d\bar{z}_2}{|z_2|^2}. \qquad (82)
\end{aligned}$$

We decompose $\theta = \theta_1 + \theta_2$, where θ_i are of the form $f_i \cdot dz_i/z_i$. Then we have the following equality:

$$\begin{aligned}
\bigl\langle \theta, \theta \bigr\rangle \wedge \partial\overline{\partial}\chi_N = {} & \bigl\langle \theta_1, \theta_1 \bigr\rangle \cdot \psi\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi''\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2}\frac{dz_2 \cdot d\bar{z}_2}{|z_2|^2} \\
& + \bigl\langle \theta_1, \theta_2 \bigr\rangle \cdot \psi'\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi'\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2}\frac{dz_2 \cdot d\bar{z}_1}{z_2 \cdot \bar{z}_1} \\
& + \bigl\langle \theta_2, \theta_1 \bigr\rangle \cdot \psi'\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi'\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2}\frac{dz_1 \cdot d\bar{z}_2}{z_1 \cdot \bar{z}_2} \\
& + \bigl\langle \theta_2, \theta_2 \bigr\rangle \cdot \psi\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi''\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2}\frac{dz_1 \cdot d\bar{z}_1}{|z_1|^2}. \qquad (83)
\end{aligned}$$

We use the real coordinate $z_i = \exp\bigl(\sqrt{-1}x_i - y_i\bigr)$ as usual, and we use the results in the subsubsection 6.1.4. Let us estimate the following integral:

$$\begin{aligned}\int_{M_P} \bigl\langle \theta_1, \theta_1 \bigr\rangle \cdot \psi\bigl(2N^{-1}y_1\bigr) \cdot \psi''\bigl(2N^{-1}y_2\bigr) \cdot \frac{1}{N^2}\frac{dz_2 \cdot d\bar{z}_2}{|z_2|^2} \\ = \int_{M_P} \bigl|f_1\bigr|_h^2 \cdot \psi\bigl(2N^{-1}y_1\bigr) \cdot \psi''\bigl(2N^{-1}y_2\bigr) \cdot \frac{1}{N^2} dx_1 \cdot dy_1 \cdot dx_2 \cdot dy_2 \\ = \int_{\boldsymbol{R}^2_{\geq 0}} \Phi_1 \cdot \frac{1}{N^2} \cdot \psi\bigl(2N^{-1}y_1\bigr) \cdot \psi''\bigl(2N^{-1}y_2\bigr) \cdot \frac{dy_1}{(2y_1 + A)^2} \cdot dy_2. \qquad (84)\end{aligned}$$

Due to Lemma 6.9, the right hand side can be rewritten as follows:

$$\begin{aligned}\int_{\boldsymbol{R}^2_{\geq 0}} \rho_1^2 \cdot \frac{1}{N^2} \cdot \psi\bigl(2N^{-1}y_1\bigr) \cdot \psi''\bigl(2N^{-1}y_2\bigr) \cdot dy_1 \cdot dy_2 \\ + \int_{\boldsymbol{R}^2_{\geq 0}} J_{4\,1} \cdot \frac{(2y_2 + A)^2}{N^2} \cdot \psi\bigl(2N^{-1}y_1\bigr) \cdot \psi''\bigl(2N^{-1}y_2\bigr) \cdot \frac{dy_1}{(2y_1 + A)^2}\frac{dy_2}{(2y_2 + A)^2}. \qquad (85)\end{aligned}$$

Since $\psi'(0) = \psi'(\infty) = 0$, the first term in (85) vanishes. We have the boundedness of the functions $N^{-2} \cdot (2y_2 + A)^2 \cdot \psi''\bigl(2N^{-1}y_2\bigr)$ independently of N, and the supports of the functions go to infinity when $N \to \infty$. Thus the second term converges to 0 when $N \to \infty$. Namely we obtain the following convergence:

$$\lim_{N\to\infty} \bigl\langle \theta_1, \theta_1 \bigr\rangle \cdot \psi\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi''\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2} \cdot \frac{dz_2 \cdot d\bar{z}_2}{|z_2|^2} = 0.$$

Similarly, we obtain the following:

$$\lim_{N\to\infty} \bigl\langle \theta_2, \theta_2 \bigr\rangle \cdot \psi''\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2} \cdot \frac{dz_1 \cdot d\bar{z}_1}{|z_1|^2} = 0.$$

It is easy to check the following estimate:

$$\bigl\langle \theta_1, \theta_2 \bigr\rangle \cdot \frac{dz_2}{z_2} \cdot \frac{d\bar{z}_1}{\bar{z}_1} = O\Bigl(\bigl|f_1\bigr| \cdot \bigl|f_2\bigr| \cdot dx_1 \cdot dy_1 \cdot dx_2 \cdot dy_2\Bigr).$$

We put $M_N := \bigl\{(z_1, z_2) \,\big|\, N/2 \leq -\log|z_i|^2 \leq 2{\cdot}N/3\bigr\}$. Then the support of the function $\prod_{i=1,2} \psi'\bigl(-N^{-1}\log|z_i|^2\bigr)$ is contained in M_N. Hence we obtain the following estimate:

$$\begin{aligned}\int \bigl\langle \theta_1, \theta_2 \bigr\rangle \cdot \psi'\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi'\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2} \cdot \frac{dz_2 \cdot d\bar{z}_1}{z_2 \cdot \bar{z}_1} = O\left(\int_{M_N} \bigl|f_1\bigr| \cdot \bigl|f_2\bigr| \cdot \frac{d\mu_1}{N^2}\right) \\ = O\left(\left(\int_{M_N} \bigl|f_1\bigr|^2 \frac{d\mu_1}{N^2}\right)^{1/2} \cdot \left(\int_{M_N} \bigl|f_2\bigr|^2 \frac{d\mu_1}{N^2}\right)^{1/2}\right). \qquad (86)\end{aligned}$$

Here we put $d\mu_1 = dx_1 \cdot dy_1 \cdot dx_2 \cdot dy_2$. Due to Lemma 6.9, we have the following inequalities:

$$\int_{M_N} \bigl|f_1\bigr|^2 \cdot \frac{d\mu}{N^2} \leq \int_{N/2}^{2N/3} dy_1 \int_{N/2}^{2N/3} dy_2 \cdot \frac{\rho_1^2}{16 \cdot N^2} + \int_{N/2}^{2N/3} \frac{dy_1}{(2y_1 + A)^2} \int_{N/2}^{2N/3} \frac{dy_2}{N^2} \cdot J_{4\,i}.$$

The second term in the right hand side converges to 0 when $N \to \infty$, due to the integrability of $J_{4\,i}$ with respect to the measure $(2y_1 + A)^{-2} \cdot (2y_2 + A)^{-2} \cdot dy_1 \cdot dy_2$. The first term is as follows:

$$\frac{\rho_i^2}{16N^2} \cdot \frac{N^2}{6^2} = \frac{\rho_i^2}{16 \times 6^2}.$$

Hence we obtain the boundedness of the following integrals, independently of N:

$$\int \bigl\langle \theta_1, \theta_2 \bigr\rangle \cdot \psi'\bigl(-N^{-1}\log|z_1|^2\bigr) \cdot \psi'\bigl(-N^{-1}\log|z_2|^2\bigr) \cdot \frac{1}{N^2}\frac{dz_2 \cdot d\bar{z}_1}{z_2 \cdot \bar{z}_1}.$$

Similarly we obtain the boundedness of the following, independent of N:

$$\int \big\langle \theta_2, \theta_1 \big\rangle \cdot \psi'\big(-N^{-1} \log |z_1|^2\big) \cdot \psi'\big(-N^{-1} \log |z_2|^2\big) \cdot \frac{1}{N^2} \frac{dz_1 \cdot d\bar{z}_2}{z_1 \cdot \bar{z}_2}.$$

Hence we obtain the boundedness of $\int_{M_P} \big\langle \theta, \theta \big\rangle \cdot \partial\overline{\partial}\chi_N$, independently of N.

Thus the proof of Lemma 6.10 is accomplished. ∎

6.3 Pluri-harmonicity

6.3.1 Statement and some reduction

Proposition 6.1 *The harmonic metric h is pluri-harmonic.*

We use the Bochner type formula in Proposition 2.2. To show Proposition 6.1, we have only to show the following vanishing of the limit:

$$\int_{X-D} d\big\langle \overline{\partial}\theta, \theta - \theta^{\dagger} \big\rangle = \lim_{N \to \infty} \int_{X-D} \chi_N \cdot d\big\langle \overline{\partial}\theta, \theta - \theta^{\dagger} \big\rangle = \lim_{N \to \infty} \int_{X-D} -d\chi_N \wedge \big\langle \overline{\partial}\theta, \theta - \theta^{\dagger} \big\rangle = 0.$$

We have only to see the vanishing of the limit of the integrals over $N_i^{\circ} - D_i^{\circ}$ and $M_P \setminus D$.

6.3.2 On $N_{i\,j}^{\circ} \setminus D$ and $N_i^{\circ} \setminus D$

We use the orthogonal decomposition $\theta = \theta_1 + \theta_2$ as in the subsubsection 6.2.2. Namely we have the following:

$$\theta_1 = \frac{1}{4} H^{-1} \cdot dH\big(\partial_\phi - \sqrt{-1} J \cdot \partial_\phi\big) \cdot \big(d\phi - \sqrt{-1} J \cdot d\phi\big).$$

Hence we have the following:

$$\theta_1^{\dagger} = \frac{1}{4} H^{-1} \cdot dH\big(\partial_\phi + \sqrt{-1} J \cdot \partial_\phi\big) \cdot \big(d\phi + \sqrt{-1} J \cdot d\phi\big).$$

Therefore we have the following:

$$\theta_1 - \theta_1^{\dagger} = -\frac{\sqrt{-1}}{2} \Big(H^{-1} dH\big(J\partial_\phi\big) \cdot d\phi + H^{-1} dH\big(\partial_\phi\big) \cdot \big(s^{-1} ds + G_2\big) \Big).$$

Here we put $G_2 := J d\phi - d \log s$. We also put $G_3 := d \log |s_i| - d \log s$. We have the estimates $\big|G_i\big| = O(1)$ $(i = 2, 3)$ with respect to the Kahler metric g_1 of X. We also have the following equality:

$$d\chi_N = -\psi'\big(-N^{-1} \log |s_i|^2\big) \cdot \frac{2}{N} \cdot d \log |s_i|.$$

Therefore we have the following equality:

$$\begin{aligned} d\chi_N \wedge \big\langle \overline{\partial}\theta, \theta - \theta^{\dagger} \big\rangle = -\psi'\big(-N^{-1} \log |s_i|^2\big) \cdot \frac{\sqrt{-1}}{N} \left(\frac{ds}{s} + G_3 \right) \wedge \big\langle \overline{\partial}\theta, H^{-1} dH\big(J\partial_\phi\big) \cdot d\phi \big\rangle \\ - \psi'\big(-N^{-1} \log s^2\big) \cdot \frac{\sqrt{-1}}{N} \left(\frac{ds}{s} + G_3 \right) \wedge \big\langle \overline{\partial}\theta, H^{-1} dH\big(\partial_\phi\big) \cdot G_2 \big\rangle \\ - \psi'\big(-N^{-1} \log s^2\big) \cdot \frac{\sqrt{-1}}{N} \cdot G_3 \wedge \big\langle \overline{\partial}\theta, H^{-1} dH\big(\partial_\phi\big) \cdot s^{-1} ds \big\rangle. \quad (87) \end{aligned}$$

Due to Lemma 6.4, the first term is dominated by the following, which is integrable (see the proof of Lemma 6.5):

$$\big|\overline{\partial}\theta\big| \cdot \Big(\big|\partial_s \Psi_h\big| \cdot \big|\partial_s\big|^{-2} + s \cdot \big|\partial_\phi \Psi_h\big| \cdot \big|\partial_\phi\big|^{-2} + s^{1/2} \sum \big|\partial_{x_i} \Psi_h\big|^2 \cdot \big|\partial_{x_i}\big|^2 \Big) \cdot \mathrm{dvol}_g\,.$$

The second term of (87)is dominated by $\big|\overline{\partial}\theta\big| \cdot \big|\partial_\phi \Psi_h\big| \cdot \mathrm{dvol}_g$, which is also integrable (Corollary 6.1 and Lemma 6.10). The third term of (87) can be dominated similarly. Hence the right hand side of (87) converges 0 when $N \to \infty$, because the support of $\psi'\big(-N^{-1} \cdot \log s^2\big)$ goes to infinity.

6.3.3 On $M_P \setminus D$

We have $\theta - \theta^\dagger = \theta_1 - \theta_1^\dagger + \theta_2 - \theta_2^\dagger$. We use the real coordinate $z_i = \exp\bigl(\sqrt{-1}x_i - y_i\bigr)$, and we use the result in the subsubsection 6.1.4. We have the following formula:

$$\theta_i - \theta_i^\dagger = \frac{\sqrt{-1}}{2}\left(h^{-1}\frac{\partial h}{\partial y_i}\cdot dx_i - h^{-1}\frac{\partial h}{\partial x_i}\cdot dy_i\right). \tag{88}$$

We also have the following on M_P:

$$d\chi_N = -\psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi\bigl(2N^{-1}y_2\bigr)\cdot\frac{2\cdot dy_1}{N} - \psi\bigl(2N^{-1}y_1\bigr)\cdot\psi'\bigl(2N^{-1}y_2\bigr)\cdot\frac{2\cdot dy_2}{N}.$$

Hence we have the following formula:

$$\begin{aligned} d\chi_N\wedge\bigl\langle\overline{\partial}\theta,\theta-\theta^\dagger\bigr\rangle &= -\psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi'\bigl(2N^{-1}y_2\bigr)\cdot\frac{2\cdot dy_1}{N}\wedge\Bigl\langle\overline{\partial}\theta,\ \frac{\sqrt{-1}}{2}h^{-1}\partial_{y_1}h\cdot dx_1+\theta_2-\theta_2^\dagger\Bigr\rangle \\ &\quad - \psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi\bigl(2N^{-1}y_2\bigr)\frac{2\cdot dy_2}{N}\wedge\Bigl\langle\overline{\partial}\theta,\ \theta_1-\theta_1^\dagger+\frac{\sqrt{-1}}{2}h^{-1}\partial_{y_2}h\cdot dx_2\Bigr\rangle. \end{aligned} \tag{89}$$

We have the following estimate independently of N:

$$\psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi\bigl(2N^{-1}y_2\bigr)\cdot\frac{2\cdot dy_1}{N}\wedge\Bigl\langle\overline{\partial}\theta,\ \frac{\sqrt{-1}}{2}h^{-1}\partial_{y_1}h\cdot dx_1\Bigr\rangle = O\Bigl(\bigl|\overline{\partial}\theta\bigr|\cdot\bigl|\partial_{y_1}\Psi_h\bigr|\cdot(2y_1+A)\cdot\mathrm{dvol}_g\Bigr).$$

Note $\overline{\partial}\theta$ and $\bigl|\partial_{y_1}\Psi_h\bigr|\cdot(2y_1+A)$ are L^2 with respect to the measure dvol_g (Lemma 6.7 and Lemma 6.10).

We have the following estimate, independently of N:

$$-\psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi\bigl(2N^{-1}y_2\bigr)\cdot\frac{2}{N}dy_1\wedge\bigl\langle\overline{\partial}\theta,\theta_2-\theta_2^\dagger\bigr\rangle = O\Bigl(\psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi\bigl(2N^{-1}y_2\bigr)\cdot\bigl|\overline{\partial}\theta\bigr|\cdot\bigl|\theta_2-\theta_2^\dagger\bigr|\cdot\mathrm{dvol}_g\Bigr). \tag{90}$$

The support of $\psi'\bigl(2N^{-1}y_1\bigr)$ is contained in $\bigl\{2^{-1}N\le y_1\le 3^{-1}N\bigr\}$. Hence (90) is dominated by the following:

$$\begin{aligned} &\int dx_1\cdot dx_2\int_{N/4}^{N/3}\frac{dy_1}{(2y_1+A)^2}\int_0^{N/3}\frac{dy_2}{(2y_2+A)^2}\cdot\bigl|\overline{\partial}\theta\bigr|\cdot\bigl|\theta_2-\theta_2^\dagger\bigr| \\ &\qquad\le\left(\int dx_1\cdot dx_2\int_{N/4}^{N/3}\frac{dy_1}{(2y_1+A)^2}\int_0^{N/3}\frac{dy_2}{(2y_2+A)^2}\cdot\bigl|\overline{\partial}\theta\bigr|^2\right)^{1/2} \\ &\qquad\qquad\times\left(\int dx_1\cdot dx_2\int_{N/4}^{N/3}\frac{dy_1}{(2y_1+A)^2}\int_0^{N/3}\frac{dy_2}{(2y_2+A)^2}\cdot\bigl|\theta_2-\theta_2^\dagger\bigr|^2\right)^{1/2}. \end{aligned} \tag{91}$$

Due to the integrability of $\bigl|\overline{\partial}\theta\bigr|^2$, the first term in the right hand side goes to 0 when $N\to\infty$. The square of the second term is dominated by the following, due to (88), Lemma 6.7 and Lemma 6.9:

$$\int_{N/4}^{N/3}\frac{dy_1}{(2y_1+A)^2}\int_0^{N/3}\frac{dy_2}{(2y_2+A)^2}\left(\rho_2^2\cdot(2y_2+A)^2+J\right).$$

Here J denotes an integrable function with respect to the measure $(2y_1+A)^{-2}\cdot(2y_2+A)^{-2}\cdot dy_1\cdot dy_2$. The contributions of J go to 0 when $N\to\infty$, due to the integrability of J. On the other hand, $\int_{N/4}^{N/3}(2y_1+A)^{-2}\cdot dy_1\times\int_0^{N/3}\rho_2^2\cdot dy_2$ are bounded independently of N. Thus the second term in the right hand side of (91) is bounded. Hence the right hand side of (91) converges to 0 when $N\to\infty$. Thus we obtain the following convergence:

$$\lim_{N\to\infty}\int_{M_P}-\psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi'\bigl(2N^{-1}y_2\bigr)\frac{2\cdot dy_1}{N}\wedge\Bigl\langle\overline{\partial}\theta,\ \frac{\sqrt{-1}}{2}h^{-1}\partial_{y_1}h\cdot dx_1+\theta_2-\theta_2^\dagger\Bigr\rangle=0.$$

Similarly, we obtain the following convergence:

$$\lim_{N\to\infty}\int_{M_P} -\psi'\bigl(2N^{-1}y_1\bigr)\cdot\psi\bigl(2N^{-1}y_2\bigr)\frac{2\cdot dy_2}{N}\wedge\Bigl\langle\overline{\partial}\theta,\ \theta_1-\theta_1^{\dagger}+\frac{\sqrt{-1}}{2}h^{-1}\partial_{y_2}h\cdot dx_2\Bigr\rangle=0.$$

Thus we obtain the following convergence:

$$\lim_{N\to\infty}\int_{M_P} d\chi_N\wedge\bigl\langle\overline{\partial}\theta,\theta-\theta^{\dagger}\bigr\rangle=0.$$

Hence the proof of Proposition 6.1 is accomplished. ∎

6.4 Tameness and pure imaginary property

Let P be a point of $D_i\cap D_j$. For simplicity, we consider the case $i=1$, $j=2$. Recall the integrability of J_{102} on $M_{P\,1}$ with respect to the measure $(y_2+A)^{-2}dy_1\cdot dx_2\cdot dy_2$ in the subsubsection 6.1.4. (see the page 49). For any point $Q\in\overline{\Delta}^*$, we put $M_{P\,1\,Q}:=\bigl\{(y_1,Q)\,\big|\,y_1\in\boldsymbol{R}_{\geq 0}\bigr\}\subset M_{P\,1}$. Then the restrictions of J_{102} to $M_{P\,1\,Q}$ are integrable with respect to the measure dy_1 for almost all $Q\in\overline{\Delta}^*$ by the theorem of Fubini.

On the other hand, we obtain the integrability of the restriction of $\bigl|\partial_{y_1}\Psi_h\bigr|^2\cdot\bigl|\partial_{y_1}\bigr|^{-2}$ to $\overline{\Delta}^*\times Q$ with respect to the measure $(2y_1+A)^{-2}\cdot dx_1\cdot dy_1$ for almost every $Q\in\overline{\Delta}^*$, from Lemma 6.7. Thus we obtain the following lemma.

Lemma 6.12 *For almost every $Q\in\overline{\Delta}^*$, there exists an integrable function J_Q on $\overline{\Delta}^*$ with respect to $\mathrm{dvol}_Q=(2y_1+A)^{-2}\cdot dx_1\cdot dy_1$, such that the following holds:*

$$\int_{T(R)\times Q}\bigl|\theta_1\bigr|^2\cdot\mathrm{dvol}_Q\leq\int_{T(R)\times Q}\left(\frac{\rho_1^2}{4\pi^2}\cdot(2y_1+A)^2+J_Q\right)\cdot\mathrm{dvol}_Q\,.$$

Here we put $T(R):=\bigl\{z\in\boldsymbol{C}\,\big|\,0\leq-\log|z|\leq R\bigr\}$. ∎

Corollary 6.3 *For almost every $Q\in\overline{\Delta}^*$, the restriction $h_{|\overline{\Delta}^*\times Q}$ is tame and pure imaginary.*

Proof It follows from Lemma 6.12 and Proposition 3.1. ∎

Let W be a compact subregion contained in $U_P\cap D_1$, and we put $Y:=\overline{\Delta}\times W$.

Lemma 6.13 *The restriction $h_{|Y\setminus D}$ is tame and pure imaginary.*

Proof On $Y\setminus D$, we describe θ as follows:

$$\theta=f\cdot\frac{dz_2}{z_2}+g\cdot dz_1.$$

Then we can easily show $\det(t-f)$ is holomorphic on Y, due to Lemma 2.4 and Corollary 6.3. Since the roots of the polynomial $\det(t-f)_{|Q}$ are pure imaginary for almost every $Q\in W$ due to Corollary 6.3, the roots of the polynomial $\det(t-f)_{|Q}$ are pure imaginary for every $Q\in W$.

Let us consider $\det(t-g)$. We have $|g|^2=O\bigl(\bigl|\partial_{z_1}\Psi_h\bigr|^2\bigr)$. By using the maximum principle for the family of tame pure imaginary harmonic bundles (Lemma 3.18), we obtain the boundedness of $|g|$. Then we obtain the boundedness of $\det(t-g)$, which implies that $\det(t-g)$ is holomorphic. Thus we are done. ∎

Lemma 6.14 *The restriction $h_{|U_P}$ is tame and pure imaginary*

Proof Let us describe θ as follows, on U_P:

$$\theta=f_1\cdot\frac{dz_1}{z_1}+f_2\cdot\frac{dz_2}{z_2}.$$

Let us consider $\det(t-f_i)$. By using the previous consideration, we have already known that $\det(t-f_i)$ are holomorphic on $U_P-\{P\}$. Hence we obtain that they are holomorphic on U_P. ∎

Theorem 6.1 *The pluri-harmonic metric h of (E,∇) is tame and pure imaginary.*

Proof Let Q be any point of $D_i - \bigcup_{j\neq i} D_i \cap D_j$. Let W_0 be a coordinate neighbourhood of Q, and (z_1,z_2) be a coordinate of W_0 such that $z_2^{-1}(0) = W_0 \cap D_i$. Then we can take a sequence of coordinate neighbourhoods $W_0, W_1 \ldots, W_l$ such that $W_a \cap W_{a+1} \cap D_i \neq \emptyset$ and $W_l \subset U_P$ for some P. On each W_a, we develop $\theta = f^{(a)} \cdot dz_2^{(a)}/z_2^{(a)} + g^{(a)} \cdot dz_1^{(a)}$. By an inductive argument using Lemma 2.5, we can show that $\det(t - f^{(a)})$ and $\det(t-g^{(a)})$ are holomorphic. Hence we obtain that (E,∇,h) is tame and pure imaginary. ∎

Remark 6.1 *For the given proof of Theorem* 6.1*, the sets $D_i \cap \bigcup_{j\neq i} D_j$ have to contain some point. It is easy to modify the argument in the case $D_i \cap \bigcup_{j\neq i} D_j = \emptyset$ for some component D_i. For example, we have only to add some extra smooth divisor which intersects D_i. Or, we have only to take a point P in D_i, and we take a good coordinate around P (see the subsubsection* 5.3.1*).* ∎

6.5 The existence of pluri-harmonic metric for the higher dimensional projective case

Let X be a smooth projective variety over $\boldsymbol{C}$, and D be a normal crossing divisor. Let (E,∇) be a flat simple bundle on $X-D$.

Theorem 6.2 *There exists a tame pure imaginary pluri-harmonic metric h of (E,∇), which is unique up to positive constant multiplication.*

Proof We use an induction on $\dim(X)$.

Let us take a sufficiently ample bundle L of X. We have the vector space $H^0(X,L)$. We have the subspace $V_P := \{f \in H^0(X,L) \,|\, f(P) = 0\}$. We put as follows:

$$\mathbb{P} := \mathbb{P}(H^0(X,L)^\vee), \quad \mathbb{P}(P) := \mathbb{P}(V_P^\vee).$$

For any element $s \in \mathbb{P}$, we put $Y_s := s^{-1}(0)$. Let U_0 denote the subset of $\mathbb{P}$, which consists of the elements s satisfying that Y_s are smooth and that $Y_s \cap D$ are normal crossing. It is Zariski dense subset of $\mathbb{P}$. We put $U(P) = U_0 \cap \mathbb{P}(P)$, which is Zariski dense subset of $\mathbb{P}(P)$.

For any element $s \in U_0$, let $U(s)$ denote the subset of U_0, which consists of the elements s' satisfying that $Y_{s'}$ are transversal with Y_s and that $D \cap Y_s \cap Y_{s'}$ are normal crossing. We put $U_0^{(1)} := \big\{(s,s') \,|\, s \in U_0,\ s' \in U(s)\big\}$.

For any point $P \in X$ and any element $s \in U_0$, we put $U(s,P) := \big\{s' \in U_s \,|\, P \in Y_{s'}\big\}$.

Let s be any element of U_0. Due to the hypothesis of the induction, we can take a tame pure imaginary pluri-harmonic metric h_s of $(E,\nabla)_{|Y_s \setminus D}$.

Lemma 6.15 *Let (s,s') be an element of $U_0^{(1)}$. We take tame pure imaginary pluri-harmonic metrics h_s and $h_{s'}$ of $(E,\nabla)_{|Y_s\setminus D}$ and $(E,\nabla)_{|Y_{s'}\setminus D}$ respectively. Then there exists a positive constant a such that $h_{s\,|\,Y_s\cap Y_{s'}\setminus D} = a \cdot h_{s'\,|\,Y_s \cap Y_{s'}\setminus D}$.*

Proof It follows from the uniqueness up to positive constant multiplication (Proposition 3.4). Remark $\dim(Y_s \cap Y_{s'}) \geq 1$. ∎

Let us fix an element $s \in U_0$ and a tame pure imaginary pluri-harmonic metric h_s.

Lemma 6.16 *Let (s_1,s_2) be an element of $U_0^{(1)}$ such that $s_i \in U(s)$. Let us take h_{s_i} such as $h_{s_i\,|\,Y_s\cap Y_{s_i}\setminus D} = h_{s\,|\,Y_s \cap Y_{s_i}\setminus D}$. Assume $Y_s \cap Y_{s_1} \cap Y_{s_2} \setminus D \neq \emptyset$. Then we have $h_{s_1\,|\,Y_{s_1}\cap Y_{s_2}\setminus D} = h_{s_2\,|\,Y_{s_1}\cap Y_{s_2}\setminus D}$.*

Proof There exists a positive constant a such that $h_{s_1\,|\,Y_{s_1}\cap Y_{s_2}\setminus D} = a \cdot h_{s_2\,|\,Y_{s_1}\cap Y_{s_2}\setminus D}$. On $Y_s \cap Y_{s_1} \cap Y_{s_2} \setminus D$, we have $h_{s_i\,|\,Y_{s_1}\cap Y_{s_2}\cap Y_s \setminus D} = h_{s\,|\,Y_{s_1}\cap Y_{s_2}\cap Y_s\setminus D}$. Hence we obtain $a=1$. Thus we are done. ∎

Lemma 6.17 *Let Q be any point $X-D$. For any elements $s_i \in U(s,Q)$ $(i=1,2)$, there exists an elements $s_3 \in U(s,Q)$ such that $(s_i,s_3) \in U_0^{(1)}$ and $Y_s \cap Y_{s_i} \cap Y_3 \setminus D \neq \emptyset$. Moreover the set of such s_3 is Zariski dense in $U(s,Q)$.*

Proof It follows from an easy argument using Zariski density. ∎

Let s_1 be a section of $U(s,Q)$. Let us take h_{s_1} such as $h_{s_1\,|\,Y_s\cap Y_{s_1}\setminus D} = h_{s\,|\,Y_s\cap Y_{s_1}\setminus D}$. We put $h_Q := h_{s_1\,|\,Q}$.

Lemma 6.18 *We have the C^∞-hermitian metric h of (E,∇) such that the following holds:*

$$h_{|Q} = h_Q, \quad h_{|Y_s} = h_s, \quad h_{|Y_{s_1}} = h_{s_1},\ (s_1 \in U(s,Q)).$$

Proof It follows from Lemma 6.16 and Lemma 6.17. ∎

Lemma 6.19 *The metric h is pluri-harmonic metric.*

Proof We denote the corresponding $(1,0)$-form by θ (the subsubsection 2.5.3). We have only to show that $\overline{\partial}\theta = \theta^2 = 0$. Let Q be any point of $X - D$ and H be a $\boldsymbol{C}$-subspace of T_QX of codimension one. We can take $s_1 \in U(s,Q)$ such that $T_QY_{s_1} = H$. Since the restriction of h to Y_{s_1} is pluri-harmonic, we have $\overline{\partial}\theta_{|H} = \theta^2_{|H} = 0$. Then we obtain $\overline{\partial}\theta = \theta^2 = 0$. ∎

Lemma 6.20 *h is tame and pure imaginary.*

Proof Once we know the tameness, we obtain the pure imaginary property by considering the restriction of h to any Y_s. Let us show the tameness.

Let Q be a smooth point of D. Let us take a neighbourhood U of Q with a coordinate $(z_1,\ldots,z_n)$ such that $U \cap D = z_1^{-1}(0)$. We have the description:

$$\theta = f_1 \cdot \frac{dz_1}{z_1} + \sum_{j=2}^{n} g_j \cdot dz_j.$$

Let us see that the coefficients of the characteristic polynomials $\det(t - f_1)$ and $\det(t - g_j)$ are holomorphic on U.

We put $S_i = \{z_i = 0\}$. We have the naturally defined projection $\pi_2 : U \longrightarrow S_1 \cap S_2$. For any point $P \in S_1 \cap S_2$, let us consider the restriction to $\pi_2^{-1}(P)$. Here we may assume that $\pi_2^{-1}(P)$ is an intersection $Y_s \cap U$ for some s. Then we have already known that $\det(t - f_1)_{|\pi_2^{-1}(P)}$ and $\det(t - g_2)_{|\pi_2^{-1}(P)}$ are holomorphic on $\pi_2^{-1}(P)$. Then it is easy to derive that $\det(t - f_1)$ and $\det(t - g_2)$ are holomorphic on U. Similarly, we can derive that $\det(t - g_i)$ are holomorphic on U.

Let Q be any point of D. Let U be a neighbourhood of Q with coordinate $(z_1,\ldots,z_n)$ such that $D \cap U = \bigcup_{i=1}^{l}\{z_i = 0\}$. We have the description:

$$\theta = \sum_{j=1}^{l} f_j \cdot \frac{dz_j}{z_j} + \sum_{j=l+1}^{n} g_j \cdot dz_j.$$

By applying the consideration above, we have already known that $\det(t - f_j)$ and $\det(t - g_j)$ are holomorphic outside the subset of codimension two. Then we obtain that they are holomorphic on U due to Hartogs' theorem. Thus we obtain Lemma 6.20, and hence the proof of Theorem 6.2 is accomplished. ∎

7 An application

7.1 Preliminary (pull back of tame harmonic bundle)

Let X and Y be smooth projective varieties over $\boldsymbol{C}$. Let D_X and D_Y be normal crossing divisors of X and Y respectively. Let $F : X \longrightarrow Y$ be a morphism such that $F^{-1}(D_Y) \subset D_X$. Recall that we have the natural morphism $F^*\Omega_Y^{1,0}(\log D_Y) \longrightarrow \Omega_X^{1,0}(\log D_X)$.

Lemma 7.1 *Let $(E, \overline{\partial}_E, \theta, h)$ be a tame harmonic bundle on $Y - D_Y$. Then the pull back $F^*(E, \overline{\partial}_E, \theta, h)$ is a tame harmonic bundle on $X - D$. If $(E, \overline{\partial}_E, \theta, h)$ is pure imaginary, then $F^*(E, \overline{\partial}_E, \theta, h)$ is also pure imaginary.*

Proof We take a prolongment $(\tilde{E},\tilde{\theta})$ of (E,θ) (Lemma 3.1), and then the eigenvalues of the residues of θ are pure imaginary, by definition. Then we obtain the section $F^*(\tilde{\theta})\in End\bigl(F^*\tilde{E}\bigr)\otimes F^*\Omega_Y^{1,0}(\log D_Y)$. It naturally induces the regular Higgs field $\theta_1\in \mathrm{End}\bigl(F^*\tilde{E}\bigr)\otimes\Omega_X^{1,0}(\log D_X)$. The restriction of $(F^*\tilde{E},\theta_1)$ to $X-D_X$ obviously coincides with $F^*_{|X-D_X}(E,\theta)$. It implies that $F^*_{|X-D_X}(E,\theta)$ is tame.

Assume that $(E,\overline{\partial}_E,\theta,h)$ is pure imaginary. We have the irreducible decompositions $D_X=\bigcup_i D_{X,i}$ and $D_Y=\bigcup_i D_{Y,i}$. Let P be a point of $D_{X,k}-\bigcup_{j\neq k}D_{X,k}\cap D_{X,j}$. Let $D_{Y,i_1},\ldots,D_{Y,i_l}$ be the irreducible components of D_Y, which contain $F(P)$. Then the residue $\mathrm{Res}_{D_{X,k}}(\theta_1)_{|P}$ can be described as the linear combination of $\mathrm{Res}_{D_{Y,i_j}}(\tilde{\theta})_{|F(P)}$ $(j=1,\ldots,l)$ with positive integer coefficients. We also note that $\mathrm{Res}_{D_{Y,i_j}}(\tilde{\theta})_{|F(P)}$ $(j=1,\ldots,l)$ are commutative, and that their eigenvalues are pure imaginary. Thus we obtain that the eigenvalues of $\mathrm{Res}_{D_{X,k}}(\theta_1)_{|P}$ are pure imaginary. Thus we can conclude that $F^*_{|X-D_X}(E,\overline{\partial}_E,\theta,h)$ is also pure imaginary. ∎

7.2 Pull back of semisimple local system

The following theorem is the answer to a question posed by Kashiwara.

Theorem 7.1 *Let X and Y be irreducible quasi projective varieties over $\boldsymbol{C}$. Let L be a semisimple local system on Y. Let $F:X\longrightarrow Y$ be a morphism. Then $F^{-1}(L)$ is also semisimple.*

Proof We may assume that X and Y are smooth. By a standard argument, we can take smooth projective varieties $\overline{X}$ and $\overline{Y}$ such that the following holds:

- We have the inclusions $X\subset\overline{X}$ and $Y\subset\overline{Y}$. The complements $D_{\overline{X}}:=\overline{X}-X$ and $D_{\overline{Y}}:=\overline{Y}-Y$ are normal crossing divisors.
- We have the morphism $\overline{F}:\overline{X}\longrightarrow\overline{Y}$ such that $\overline{F}(X)\subset Y$ and $\overline{F}_X=F$. Note that we have $\overline{F}^{-1}(D_{\overline{Y}})\subset D_{\overline{X}}$.

Let (E,∇) be a flat bundle on Y corresponding to L. Then we can take a tame pure imaginary pluri-harmonic metric h of (E,∇). Then $F^{-1}(E,\nabla,h)$ is a tame pure imaginary harmonic bundle on $X=\overline{X}-D_{\overline{X}}$. Hence $F^{-1}(E,\nabla)$ is semisimple. Thus we are done. ∎

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